

UNIT-II

ELECTRIC FLUX:

1837 - Michael Faraday - Experimented on Electric Field.

- Electric field around a charge can be imagined as lines of force around it.

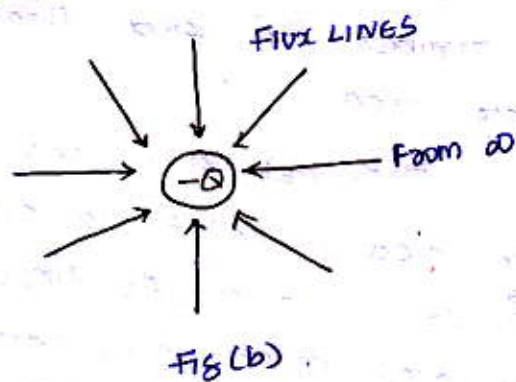
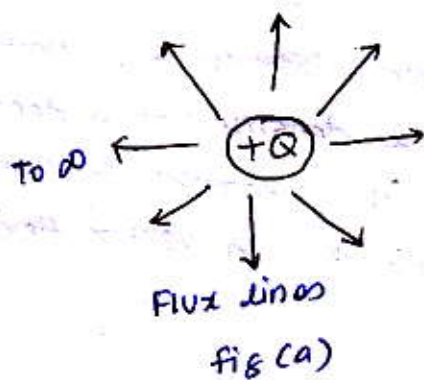
- He suggested that the electric field should be assumed to be composed of very small bunches containing a fixed number of electric lines of force. Such a bunch or closed area is called a tube of flux. [The total number of tubes of flux in any particular electric field is called as the Electric flux.]

[The total number of lines of force in any particular electric field is called the electric flux.] It is represented by the symbol ψ . Similar to the charge, unit of electric flux is also Coulomb C.

PROPERTIES OF FLUX LINES:

Electric flux - The lines of force, around a charge.

1. The flux lines start from +ve charge and terminate on the negative charge.
2. If -ve charge is absent, then the flux lines will terminate at infinity as shown in fig (a). While in absence of +ve charge, the electric flux terminates on the negative charge from infinity. This is shown in fig (b).



3. There are more number of lines. i.e. crowding of lines if electric field is stronger.

4. The flux lines are independent of the medium in which charges are placed.

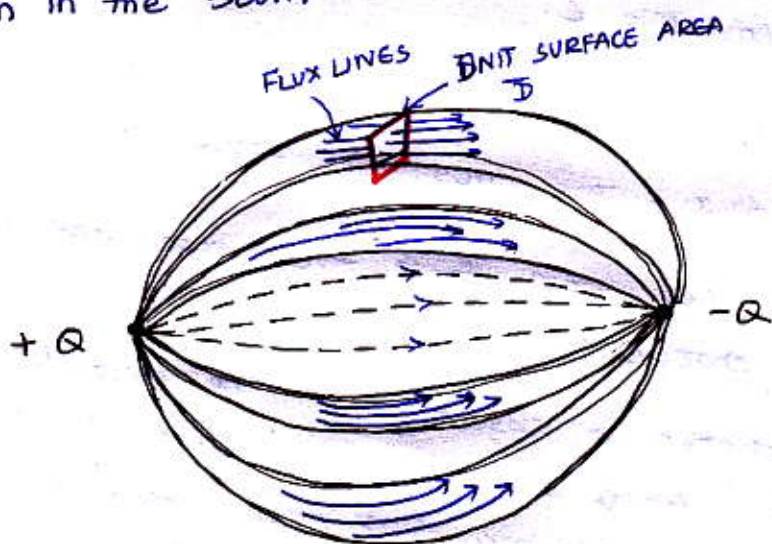
5. If the charge on a body is $\pm Q$ Coulombs, then the total number of lines originating or terminating on it is also Q .

$$\therefore \text{Electric flux } \phi = Q \text{ Coulombs}$$

The electric flux is also called DISPLACEMENT FLUX.

ELECTRIC FLUX DENSITY (\vec{D})

Consider the two point charges as shown in the figure below. The flux lines originating from positive charge and terminating at negative charge as shown in the form of tubes.



Consider a unit surface area as shown in figure. The number of flux lines are passing thro' this surface area.

The net flux passing normal through the unit surface area is called the electric flux density. It is denoted as \vec{D} . It has a specific direction which is normal to the surface area under consideration hence it is a vector field.

Consider a sphere with a charge Q placed at its centre. There are no other charges present around. The total flux distributes radially around the charge is $\psi = Q$. This flux distributes uniformly over the surface of the sphere.

$$\psi = \text{Total flux}$$

$$S = \text{Total surface area of sphere}$$

Then Electric flux density is defined as

$$D = \frac{\psi}{S} \text{ in magnitude}$$

ψ is measured in coulombs

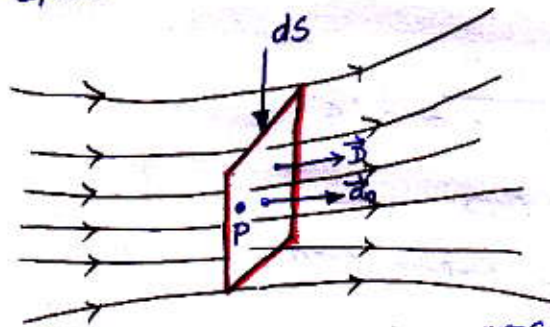
S is measured in sq. meters

$\therefore D$ unit is C/m^2 .

'D' is also called as displacement flux density (or) displacement density.

VECTOR FORM OF ELECTRIC FLUX DENSITY:

Consider the flux distribution, due to a certain charge in the free space as shown in figure below.



Consider the differential surface area ds at point P . The flux crossing through this differential area is $d\psi$. The direction of \vec{D} is same as that of direction of flux lines at that point.

The differential area and flux lines are at right angles to each other at point P . Hence the direction of \vec{D} is also normal to the surface area ds . Near point P , all the lines of flux $d\psi$ are having same direction as that of \vec{d}_n .

Hence the flux density \vec{D} at the point P can be represented in the vector form as

$$\vec{D} = \frac{d\psi}{ds} \vec{a}_n \text{ C/m}^2$$

$d\psi \rightarrow$ Total flux lines crossing normal thro' the differential area ds .

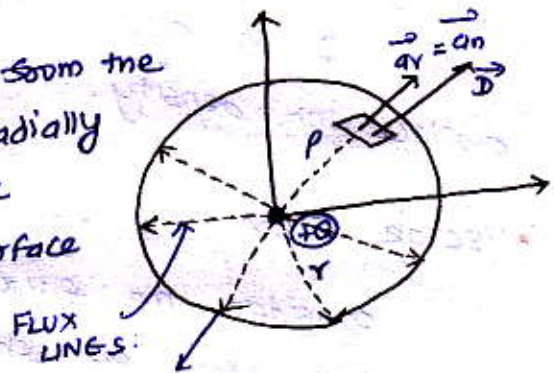
$ds \rightarrow$ Differential surface area

$\vec{a}_n \rightarrow$ unit vector in the direction normal to the differential surface area.

\vec{D} due to a point charge Q

consider a point charge $+Q$ placed at the centre of the imaginary sphere of radius r . This is shown in the figure.

The flux lines originating from the point charge $+Q$ are directed radially outwards. The magnitude of flux density at any point on the surface is.



$$|\vec{D}| = \frac{\text{Total flux } \psi}{\text{Total surface area 'S'}}$$

$$\psi = Q = \text{total flux}$$

$$S = 4\pi r^2 = \text{Total surface area}$$

$$\boxed{|\vec{D}| = \frac{Q}{4\pi r^2}}$$

The unit vector directed radially outwards and normal to the surface at any point on the sphere is $\vec{a}_n = \vec{a}_r$. Thus in vector form, Electric flux density at a point which is at a distance of r , from the point charge $+Q$ is given by

$$\boxed{\vec{D} = \frac{Q}{4\pi r^2} \vec{a}_r \text{ C/m}^2}$$

RELATIONSHIP BETWEEN \vec{D} & \vec{E}

We know that, The Electric field Intensity \vec{E} at a distance of r , from a Point charge $+Q$ is given by

$$\vec{E} = \frac{Q}{4\pi\epsilon_0 r^2} \vec{a}_r \quad \text{--- (1)}$$

& Electric flux density (\vec{D}) is given by

$$\vec{D} = \frac{Q}{4\pi r^2} \vec{a}_r \quad \text{--- (2)}$$

(2) \div (1) we get

$$\frac{\vec{D}}{\vec{E}} = \frac{[Q/4\pi r^2] \vec{a}_r}{[Q/4\pi\epsilon_0 r^2] \vec{a}_r} = \epsilon_0$$

$$\vec{D} = \epsilon_0 \vec{E} \quad \text{--- for free space}$$

Thus \vec{D} & \vec{E} are related thro' the Permittivity. If the medium in which charge is located in free space is other than free space having relative Permittivity ϵ_r , then

$$\vec{D} = \epsilon_0 \epsilon_r \vec{E} = \epsilon \vec{E}$$

ELECTRIC FLUX DENSITY FOR VARIOUS CHARGE DISTRIBUTIONS:

LINE CHARGE:

consider a line charge having uniform charge density of ρ_L C/m. Then the total charge along the line is given by,

$$Q = \int_L \rho_L dl$$

$$\vec{D} = \frac{Q}{4\pi r^2} \vec{a}_r = \frac{\int_L \rho_L dl}{4\pi r^2} \vec{a}_r$$

If the line charge is infinite then \vec{E} is derived as,

$$\vec{E} = \frac{\rho_L}{2\pi\epsilon_0 r} \vec{a}_r \quad \& \quad \vec{D} = \epsilon_0 \vec{E} = \frac{\rho_L}{2\pi r} \vec{a}_r$$

SURFACE CHARGE:

Consider a sheet of charge having uniform charge density of $\rho_s \text{ C/m}^2$. Then the total charge on the surface is given by,

$$Q = \int_S \rho_s \, ds$$

$$\therefore \vec{D} = \frac{Q}{4\pi r^2} \vec{a}_r$$

$$\vec{D} = \frac{\int_S \rho_s \, ds}{4\pi r^2} \vec{a}_r$$

The integration is over the surface S and is double integral.

If the sheet of charge is infinite then \vec{E} is derived as

$$\vec{E} = \frac{\rho_s}{2\epsilon_0} \vec{a}_n$$

$$\therefore \vec{D} = \epsilon_0 \vec{E}$$

$$\vec{D} = \frac{\rho_s}{2} \vec{a}_n$$

VOLUME CHARGE

Consider a charge enclosed by a volume, with a uniform charge density $\rho_v \text{ C/m}^3$. Then the total charge enclosed by the volume is given by,

$$Q = \int_{Vol} \rho_v \, dv \quad E = \frac{\int_{Vol} \rho_v \, dv}{4\pi\epsilon_0 r^2} \vec{a}_r$$

$$\vec{D} = \epsilon_0 E = \frac{\int_{Vol} \rho_v \, dv}{4\pi r^2} \vec{a}_r$$

PROBLEM:

Find \vec{D} in Cartesian co-ordinates at Point $(b, 8, -10)$ due to

- a) a Point charge of 40mC at the origin.
- b) a uniform line charge of $P_L = 40\text{mC/m}$ on the z -axis
- & c) a uniform surface charge density $P_s = 57.2\text{NC/m}^2$ on the Plane $x=12\text{m}$

a) A Point charge of 40mC at the origin z

$P(b, 8, -10)$ & $O(0, 0, 0)$

$$\therefore \vec{r} = (b-0)\vec{a}_x + (8-0)\vec{a}_y + (-10-0)\vec{a}_z$$

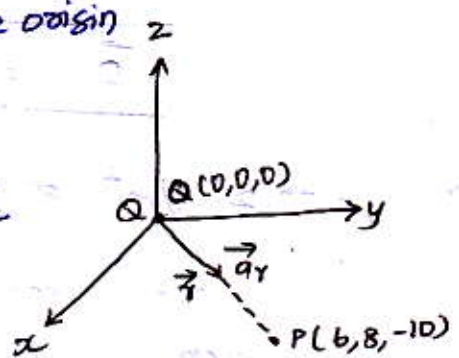
$$\vec{r} = 6\vec{a}_x + 8\vec{a}_y - 10\vec{a}_z$$

$$\therefore |\vec{r}| = \sqrt{6^2 + 8^2 + 10^2} = \sqrt{200}$$

$$\therefore \vec{a}_r = \frac{\vec{r}}{|\vec{r}|} = \frac{6\vec{a}_x + 8\vec{a}_y - 10\vec{a}_z}{\sqrt{200}}$$

$$\therefore \vec{D} = \frac{Q}{4\pi r^2} \vec{a}_r = \frac{40 \times 10^{-3}}{4\pi (\sqrt{200})^2} \left\{ \frac{6\vec{a}_x + 8\vec{a}_y - 10\vec{a}_z}{\sqrt{200}} \right\}$$

$$\vec{D} = 6.752 \times 10^{-6} \vec{a}_x + 9.003 \times 10^{-6} \vec{a}_y - 11.254 \times 10^{-6} \vec{a}_z \text{ C/m}^2$$



(b) $P_L = 40\text{mC/m}$ along z -axis.

The charge is infinite hence,

$$\vec{E} = \frac{P_L}{2\pi\epsilon_0 r} \vec{a}_r$$

As the charge is along z -axis there cannot be any component of \vec{E} along z -direction.

Consider a point on the line charge $(0, 0, z)$ and $P(b, 8, -10)$. But while obtaining \vec{r} do not consider z co-ordinate as \vec{E} and \vec{D} have no \vec{a}_z component.

$$\therefore \vec{r} = (b-0)\vec{a}_x + (8-0)\vec{a}_y = 6\vec{a}_x + 8\vec{a}_y$$

$$|\vec{y}| = \sqrt{6^2 + 8^2} = 10$$

$$\vec{a}_y = \frac{6\vec{a}_x + 8\vec{a}_y}{10}$$

$$\vec{E} = \frac{P_L}{2\pi\epsilon_0(10)} \left[\frac{6\vec{a}_x + 8\vec{a}_y}{10} \right]$$

$$\vec{D} = \epsilon_0 \vec{E} = \frac{P_L}{2\pi \times 10} \left[\frac{6\vec{a}_x + 8\vec{a}_y}{10} \right]$$

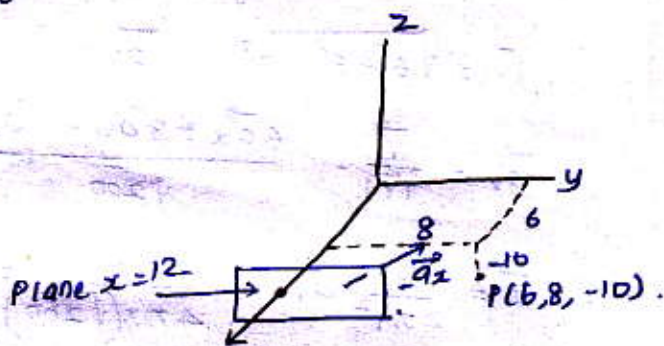
$$\vec{D} = 3.819 \times 10^{-7} \vec{a}_x + 5.092 \times 10^{-7} \vec{a}_y \text{ C/m}^2$$

c) $P_s = 57.2 \text{ NC/m}^2$ on the plane $x=12$

The sheet of charge is infinite over the plane $x=12$ which is \parallel to yz plane. The unit vector normal to this plane is $\vec{a}_n = \vec{a}_x$

$$\therefore E = \frac{P_s}{2\epsilon_0} \vec{a}_n$$

$$= \frac{P_s}{2\epsilon_0} (-\vec{a}_x)$$



The Point x
P is back side of the plane hence $\vec{a}_n = -\vec{a}_x$ as shown in figure.

$$\therefore \vec{E} = -\frac{P_s}{2\epsilon_0} \vec{a}_x$$

$$\vec{D} = \epsilon_0 \vec{E} = -\frac{P_s}{2} \vec{a}_x = 28.6 \times 10^{-6} \vec{a}_x \text{ C/m}^2$$

GAUSS'S LAW:

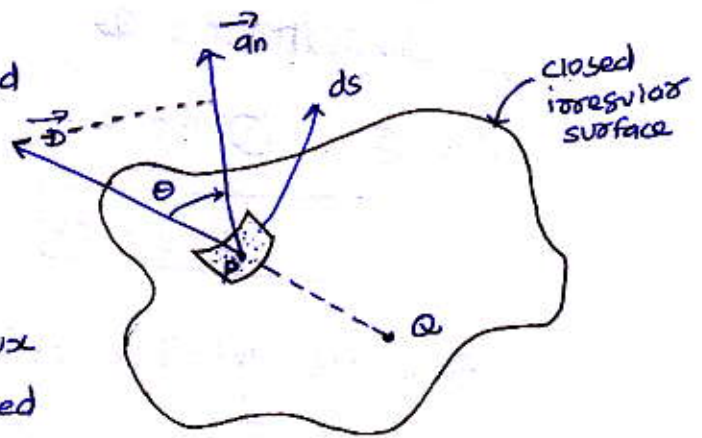
The electric flux passing thro' any closed surface is equal to the total charge enclosed by that surface.

$$\begin{aligned} \frac{d\phi}{dA} \cos\theta &= \frac{1}{101} \\ \int \frac{d\phi}{dA} \cos\theta &= \frac{2\pi}{101} \end{aligned}$$

MATHEMATICAL REPRESENTATION OF GAUSS'S LAW:

Consider any object of irregular shape as shown in the figure.

The total charge enclosed by the irregular closed surface is Q Coulombs. It may be in any form of distribution. Hence the total flux that has to pass thro' the closed surface is Q .



Consider a small differential surface ds at point P . As the surface is irregular, the direction of \vec{D} as well as its magnitude is going to change from point to point on the surface.

The surface ds under consideration can be represented in the vector form in terms of its area and direction normal to the surface at the point.

$$\therefore \vec{ds} = ds \vec{a}_n$$

where \vec{a}_n is unit vector normal to the surface ds at point P .

The flux density at point P is \vec{D} and its direction is such that it makes an angle θ with the normal direction at point P .

The flux $d\psi$ passing through the surface ds is the product of the component of \vec{D} in the direction normal to the ds and ds .

$$\text{i.e.} = \boxed{d\psi = \vec{D}_n ds} \quad \text{--- (1) } [\because \psi = D_s ; \therefore d\psi = D ds]$$

where D_n is the component of \vec{D} in the direction of normal to the surface ds .

from the figure

$$D_n = |D| \cos \theta \quad \text{--- (2)}$$

~~Q. 1. 2. 3. 4. 5. 6. 7. 8. 9. 10. 11. 12. 13. 14. 15. 16. 17. 18. 19. 20. 21. 22. 23. 24. 25. 26. 27. 28. 29. 30. 31. 32. 33. 34. 35. 36. 37. 38. 39. 40. 41. 42. 43. 44. 45. 46. 47. 48. 49. 50. 51. 52. 53. 54. 55. 56. 57. 58. 59. 60. 61. 62. 63. 64. 65. 66. 67. 68. 69. 70. 71. 72. 73. 74. 75. 76. 77. 78. 79. 80. 81. 82. 83. 84. 85. 86. 87. 88. 89. 90. 91. 92. 93. 94. 95. 96. 97. 98. 99. 100.~~

sub (2) in (1) we get

$$d\psi = |D| \cos \theta ds \quad \text{--- (3)}$$

from the definition of dot product

$$\vec{A} \cdot \vec{B} = |A| |B| \cos \theta_{AB}$$

\therefore we can write

$$|D| ds \cos \theta = \vec{D} \cdot \vec{ds}$$

$$\therefore d\psi = \vec{D} \cdot \vec{ds} \quad \text{--- (4)}$$

[Density is mass per unit area]

This is the flux passing through the incremental surface area ds .

Hence the total flux passing through the entire closed surface is to be obtained by finding the surface integration of the equation (4)

$$\psi = \int d\psi = \oint_S \vec{D} \cdot \vec{ds} \quad \text{--- (5)}$$

\oint \rightarrow Indicates integration over the closed surface and called closed surface integral. It's a double integration. closed surface over which the integration is carried out is called GAUSSIAN SURFACE.

Now irrespective of the shape of the surface and the charge distribution, the total flux passing through the surface is the total charge enclosed by the surface.

$$\psi = \oint_S \vec{D} \cdot \vec{ds} = Q = \text{charge enclosed}$$

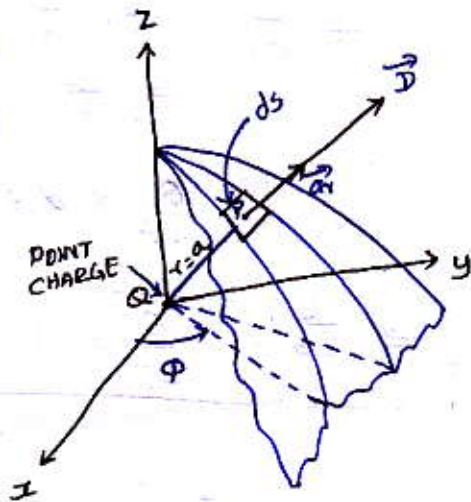
\rightarrow Mathematic representation of Gauss' law.

PROOF FOR GAUSS'S LAW:

Let, a point charge Q is located at the origin.

To determine \vec{D} and to apply Gauss's law, consider a spherical surface around Q , with centre as origin. This spherical surface is gaussian surface. The \vec{D} is always directed radially outwards along \vec{a}_r which is normal to the spherical surface at any point P on the surface. This is shown in figure below.

Consider a differential surface area ds as shown. The direction normal to the surface ds is \vec{a}_r , considering spherical co-ordinate system. The radius of the sphere is $r=a$.



The direction of \vec{D} is along \vec{a}_r which is normal to ds at any point P .

In spherical co-ordinate systems, the ds normal to radial direction \vec{a}_r is given by [Refer Unit 1 Page No: 24].

$$ds = r^2 \sin\theta \, d\theta \, d\phi \quad \text{--- (1)}$$

W.K.T. : $r = a$ --- (2)

Sub (2) in (1) we get

$$ds = a^2 \sin\theta \, d\theta \, d\phi \quad \text{--- (3)}$$

$$\therefore \vec{ds} = ds \vec{a}_n \quad \text{--- (4)}$$

& $\vec{a}_n = \vec{a}_r$

$$\therefore \vec{ds} = a^2 \sin\theta \, d\theta \, d\phi \vec{a}_r \quad \text{--- (5)}$$

Now \vec{D} due to point charge is given by,

$$\vec{D} = \frac{Q}{4\pi r^2} \vec{a}_r = \frac{Q}{4\pi a^2} \vec{a}_r \quad [\text{as } r=a]$$

$$\therefore \vec{D} \cdot d\vec{s} = |\vec{D}| |d\vec{s}| \cos \theta'$$

where θ' is the angle b/w \vec{D} and $d\vec{s}$

$$\text{where } |\vec{D}| = \frac{Q}{4\pi a^2} \text{ ---- (6)}$$

$$|d\vec{s}| = a^2 \sin \theta d\theta d\phi \text{ ---- (7)}$$

The normal to $d\vec{s}$ is \vec{a}_r while \vec{D} also acts along \vec{a}_r hence angle between $d\vec{s}$ and \vec{D} is zero (i.e) $\theta' = 0$

$$\therefore \vec{D} \cdot d\vec{s} = |\vec{D}| |d\vec{s}| \cos 0$$

$$= |\vec{D}| |d\vec{s}|$$

$$= \frac{Q}{4\pi a^2} \times a^2 \sin \theta d\theta d\phi$$

$$\vec{D} \cdot d\vec{s} = \frac{Q}{4\pi} \sin \theta d\theta d\phi \text{ ---- (8)}$$

$$\therefore \psi = \oint_S \vec{D} \cdot d\vec{s}$$

$$= \int_{\phi=0}^{2\pi} \int_{\theta=0}^{\pi} \frac{Q}{4\pi} \sin \theta d\theta d\phi$$

$$= \frac{Q}{4\pi} \left[\int_{\phi=0}^{2\pi} d\phi \int_{\theta=0}^{\pi} \sin \theta d\theta \right]$$

$$= \frac{Q}{4\pi} \left[\phi \right]_0^{2\pi} \left[-\cos \theta \right]_0^{\pi}$$

$$= \frac{Q}{4\pi} \left[2\pi - 0 \right] \left[-\cos \pi + \cos 0 \right]$$

$$= \frac{Q}{4\pi} \times 2\pi \left[-(-1) + 1 \right] \Rightarrow \frac{Q}{4\pi} \times 2\pi \times 2$$

$$\Rightarrow \boxed{\psi = Q}$$

This proves Gauss's law that Q coulombs of flux crosses the surface if Q coulombs of charge is enclosed by that surface.

APPLICATIONS OF GAUSS'S LAW:

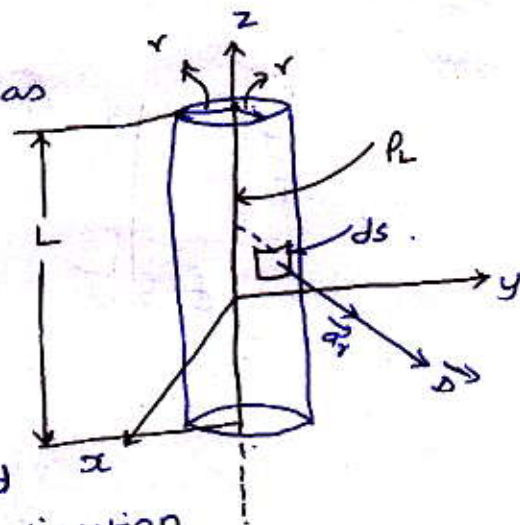
Gauss's law is used to find \vec{D} or \vec{E} due to some symmetric charge distributions like.

- (i) Point charge
- (ii) Line charge
- (iii) Surface charge
- (iv) Volume charge
- (v) Co-axial cable & so on.

GAUSS'S LAW APPLIED TO INFINITE LINE CHARGE:

Consider an infinite line charge of density ρ_L C/m lying along z-axis from $-\infty$ to $+\infty$. This is shown in figure below.

Consider the Gaussian surface as the right circular cylinder with z axis as its axis and radius r as shown in the figure. The length of the cylinder is L .



The flux density at any point on the surface is directed radially outwards i.e. in the \vec{a}_r direction according to cylindrical-co-ordinate system.

Consider a differential surface area ds as shown which is at a radial distance r from the line charge. The direction normal to ds is \vec{a}_r .

As the line charge is along z-axis, there cannot be any component of \vec{D} in the z direction. So \vec{D} has only radial component

$$\text{Now } Q = \oint \vec{D} \cdot \vec{a}_r ds$$

This integration is to be evaluated for side surface, top surface and bottom surface.

$$\therefore Q = \oint_{\text{side}} \vec{D} \cdot d\vec{s} + \oint_{\text{top}} \vec{D} \cdot d\vec{s} + \oint_{\text{bottom}} \vec{D} \cdot d\vec{s}$$

Now $\vec{D} = D_r \vec{a}_r$ has only radial component.
and $d\vec{s} = r d\phi dz \vec{a}_r$ normal to \vec{a}_r direction. [From Page No: 20
of Unit-I]

$$\vec{D} \cdot d\vec{s} = D_r \cdot r d\phi dz [\vec{a}_r \cdot \vec{a}_r]$$

$$\vec{D} \cdot d\vec{s} = D_r r d\phi dz \quad [\because \vec{a}_r \cdot \vec{a}_r = 1]$$

Note D_r is constant over the side surface

As \vec{D} has only radial component and no component along \vec{a}_z and $-\vec{a}_z$ hence integrations over top and bottom surfaces is zero.

$$\therefore \oint_{\text{top}} \vec{D} \cdot d\vec{s} = \oint_{\text{bottom}} \vec{D} \cdot d\vec{s} = 0$$

$$Q = \oint_{\text{side}} \vec{D} \cdot d\vec{s} = \oint_{\text{side}} D_r r d\phi dz$$

$$= D_r r \int d\phi dz$$

$$= r D_r \left[\int_{\phi=0}^{2\pi} d\phi \int_{z=0}^L dz \right]$$

$$= r D_r \left[[\phi]_0^{2\pi} [z]_0^L \right]$$

$$Q = 2\pi r D_r L$$

$$D_r = \frac{Q}{2\pi r L}$$

$$\therefore \vec{D} = D_r \vec{a}_r = \frac{Q}{2\pi r L} \vec{a}_r$$

$$\text{But } \frac{Q}{L} = P_L \text{ C/m.}$$

$$\therefore \vec{D} = \frac{P_L}{2\pi r} \vec{a}_r \text{ C/m}^2$$

$$\& E = \frac{\vec{D}}{\epsilon_0} = \frac{P_L}{2\pi \epsilon_0 r} \vec{a}_r \text{ V/m}$$

GAUSS'S LAW APPLIED TO INFINITE SHEET OF CHARGE:

Consider the infinite sheet of charge of uniform charge density ρ_s C/m², lying in the $z=0$ plane. i.e. xy plane as shown in fig.

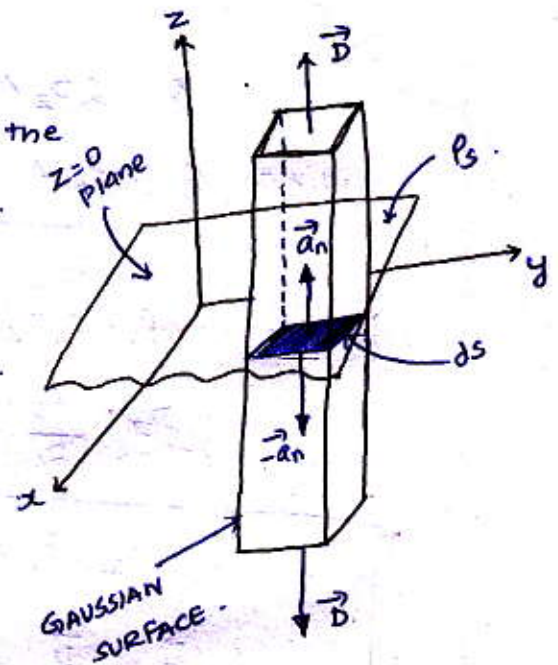
consider a rectangular box as a Gaussian surface which is cut by the sheet of charge to give $ds = dx dy$.

\vec{D} acts normal to the plane

i.e. $\vec{a}_n = \vec{a}_z$ and $-\vec{a}_n = -\vec{a}_z$ direction.

Hence $\vec{D} = 0$ in x and y directions.

Hence the charge enclosed can be written as



$$Q = \oint_S \vec{D} \cdot d\vec{s} = \oint_{\text{sides}} \vec{D} \cdot d\vec{s} + \oint_{\text{top}} \vec{D} \cdot d\vec{s} + \oint_{\text{bottom}} \vec{D} \cdot d\vec{s}$$

BUT $\oint_{\text{sides}} \vec{D} \cdot d\vec{s} = 0$ as \vec{D} has no component in x and y direction.

Now $\vec{D} = D_z \vec{a}_z$ for top surface

and $d\vec{s} = dx dy \vec{a}_z$ [From Page no: 16 of unit - I]

$$\therefore \vec{D} \cdot d\vec{s} = D_z dx dy (\vec{a}_z \cdot \vec{a}_z)$$

$$\vec{D} \cdot d\vec{s} = D_z dx dy$$

For bottom surface

$$\vec{D} = D_z (-\vec{a}_z)$$

$$d\vec{s} = dx dy (-\vec{a}_z)$$

$$\therefore \vec{D} \cdot d\vec{s} = D_z dx dy (-\vec{a}_z \cdot -\vec{a}_z)$$

$$\vec{D} \cdot d\vec{s} = D_z dx dy$$

$$\therefore Q = \oint_{\text{top}} D_z dx dy + \oint_{\text{bottom}} D_z dx dy$$

$$\text{Let } \oint_{\text{top}} dx dy = \oint_{\text{bottom}} dx dy = A = \text{Area of surface}$$

$$P_s = \frac{Q}{A}$$

$$\therefore Q = 2 D_z A$$

$$\text{W.K.T } Q = P_s A \quad \rightarrow \text{Surface charge density}$$

$$\therefore P_s = 2 D_z$$

$$D_z = \frac{P_s}{2}$$

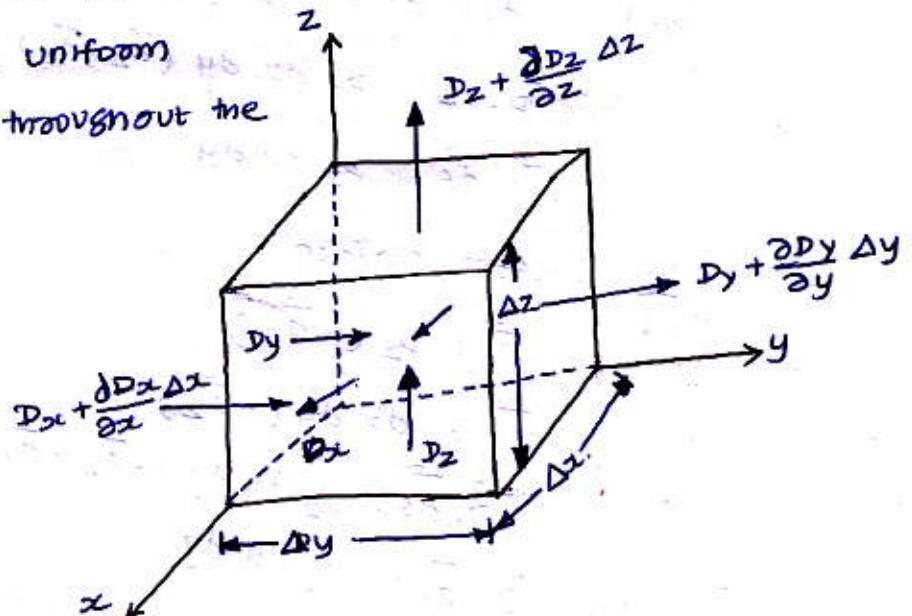
$$\therefore \vec{D} = D_z \cdot \vec{a}_z = \frac{P_s}{2} \vec{a}_z \text{ C/m}^2$$

$$\therefore \vec{E} = \frac{\vec{D}}{\epsilon_0} = \frac{P_s}{2 \epsilon_0} \vec{a}_z \text{ V/m}$$

GAUSS'S LAW APPLIED OF DIFFERENTIAL VOLUME ELEMENT.

[POINT FORM OF GAUSS'S LAW] (OR) GAUSS'S LAW IN DIFFERENTIAL FORM

Consider a small volume $\Delta V = \Delta x \Delta y \Delta z$ in Cartesian system. Here Δx , Δy and Δz are the edges of this small volume in the direction of x , y , z axes respectively. Assume uniform charge density P_v throughout the volume.



Now consider this volume is placed in an electric field with the flux density \vec{D} given by

$$\vec{D} = D_x \vec{a}_x + D_y \vec{a}_y + D_z \vec{a}_z \quad (\text{C/m}^2)$$

Here D_x , D_y and D_z are the flux densities at $x=0$, $y=0$ and $z=0$ planes respectively as shown in fig. In order to derive point form of Gauss law, we use integral form of Gauss's law.

$$\oint \vec{D} \cdot d\vec{s} = Q_{\text{enclosed}} = \int_V \rho_v dv.$$

To obtain RHS.

$$\begin{aligned} Q_{\text{enclosed}} &= \int_V \rho_v dv = \int_0^{\Delta z} \int_0^{\Delta y} \int_0^{\Delta x} \rho_v dx dy dz \\ &= \rho_v \Delta x \Delta y \Delta z \end{aligned}$$

To obtain LHS:

The closed surface integral consists of six components as

$$\oint_S \vec{D} \cdot d\vec{s} = \int_{\text{back}} \vec{D} \cdot d\vec{s} + \int_{\text{front}} + \int_{\text{left}} + \int_{\text{right}} + \int_{\text{bottom}} + \int_{\text{top}}.$$

This requires knowledge of flux density \vec{D} at each surface, which is obtained as follows:

The flux density D_x is in the direction of x -axis, then the normal outward component of \vec{D} at the back face is $-\underline{D_x}$.

If the field changes b/w the back and front faces, the rate of change of D in the x direction is $\frac{\partial D_x}{\partial x}$.

The total change in D b/w back and front face is $\frac{\partial D_x}{\partial x} \cdot \Delta x$.

But the flux density at the back face is D_x
 then the normal component of D at the front face is

$$D_x + \text{change in } D \text{ from back to front} = D_x + \frac{\partial D_x}{\partial x} \Delta x.$$

Similarly, the normal component of D at the

$$\text{left side face} = -D_y$$

$$\text{Right side face} = D_y + \frac{\partial D_y}{\partial y} \Delta y$$

$$\text{bottom face} = -D_z$$

$$\text{top face} = D_z + \frac{\partial D_z}{\partial z} \Delta z.$$

Knowing D at each surface now the integrals can be solved.

$$\int_{\text{back}} \vec{D} \cdot \vec{ds} = \int_0^{\Delta z} \int_0^{\Delta y} D_x \vec{a}_x \cdot d\vec{z} dy \vec{a}_x = \cancel{D_x \Delta z \Delta y} = -D_x \Delta z \Delta y \quad \text{--- (1)}$$

$$\int_{\text{front}} \vec{D} \cdot \vec{ds} = \int_0^{\Delta z} \int_0^{\Delta y} \left(D_x + \frac{\partial D_x}{\partial x} \Delta x \right) \vec{a}_x \cdot d\vec{z} dy \vec{a}_x = \left[D_x + \frac{\partial D_x}{\partial x} \Delta x \right] \Delta y \Delta z \quad \text{--- (2)}$$

$$\int_{\text{left}} \vec{D} \cdot \vec{ds} = -D_y \Delta x \Delta z \quad \text{--- (3)}$$

$$\int_{\text{top}} \vec{D} \cdot \vec{ds} = \left(D_z + \frac{\partial D_z}{\partial z} \Delta z \right) \Delta x \Delta y \quad \text{--- (5)}$$

$$\int_{\text{right}} \vec{D} \cdot \vec{ds} = \left(D_y + \frac{\partial D_y}{\partial y} \Delta y \right) \Delta x \Delta z \quad \text{--- (4)}$$

$$\int_{\text{bottom}} \vec{D} \cdot \vec{ds} = -D_z \Delta x \Delta y \quad \text{--- (6)}$$

① + ② + ③ + ④ + ⑤ + ⑥ we get

$$\text{LHS} = \left[\frac{\partial D_x}{\partial x} + \frac{\partial D_y}{\partial y} + \frac{\partial D_z}{\partial z} \right] \Delta x \Delta y \Delta z = \text{RHS} = \rho_v \Delta x \Delta y \Delta z$$

$$\rho_v = \frac{\partial D_x}{\partial x} + \frac{\partial D_y}{\partial y} + \frac{\partial D_z}{\partial z} = \left(\frac{\partial}{\partial x} \vec{a}_x + \frac{\partial}{\partial y} \vec{a}_y + \frac{\partial}{\partial z} \vec{a}_z \right) \cdot (D_x \vec{a}_x + D_y \vec{a}_y + D_z \vec{a}_z)$$

Maxwell's First Equation

(or) Gauss's law in vector form

$$\rho_v = \nabla \cdot \vec{D} = \text{div } \vec{D} \rightarrow \text{vector operator}$$

MATHEMATICAL DEFINITION OF DIVERGENCE: $\nabla \cdot \vec{D} = \rho_v$ scalar

consider Gauss's law for the electric field in differential form.

$$\nabla \cdot \vec{D} = \rho_v \quad \text{--- (1)}$$

We wish to express $\nabla \cdot \vec{D}$ at a point in the charge region in terms of ρ_v at that point.

① $\times \Delta V$ on both sides

$$(\nabla \cdot \vec{D}) \Delta V = \rho_v \Delta V \quad \text{--- (2)}$$

where ΔV is infinitesimal volume ΔV at that point.

But w.k.t ρ_v volume charge density is (C/m^3)

and ΔV is volume in m^3 .

$\therefore \rho_v \Delta V =$ charge contained in that volume.

\therefore evaluation ② gives the charge enclosed

$$Q_{\text{enclosed}} = \rho_v \Delta V \quad \text{--- (3)}$$

According to Gauss's law for the electric field in integral form

$$Q_{\text{enclosed}} = \oint_S \vec{D} \cdot d\vec{s} \quad \text{--- (4)}$$

③

② = ④

$$\Rightarrow \oint_S \vec{D} \cdot d\vec{s} = \rho_v \Delta V \quad \text{--- (5)}$$

sub ② in ⑤ we get

$$(\nabla \cdot \vec{D}) \Delta V = \oint_S \vec{D} \cdot d\vec{s}$$

$$\Rightarrow \nabla \cdot \vec{D} = \frac{\oint_S \vec{D} \cdot d\vec{s}}{\Delta V}$$

To express $\nabla \cdot \vec{D}$ at a point, let us limit that ΔV tends to zero, at that point

$$\nabla \cdot \vec{D} = \lim_{\Delta V \rightarrow 0} \frac{\oint \vec{D} \cdot d\vec{s}}{\Delta V}$$

Thus for any vector \vec{A} , divergence is defined as follows.

$$\nabla \cdot \vec{A} = \lim_{\Delta V \rightarrow 0} \frac{\oint \vec{A} \cdot d\vec{s}}{\Delta V} = \text{divergence of } \vec{A} \text{ (or) } \text{div } \vec{A}$$

PHYSICAL SIGNIFICANCE OF DIVERGENCE:

Definition: It is defined as the outflow of vectors over the surface per unit volume as volume approaches zero.

PROPERTIES OF DIVERGENCE OF VECTOR FIELD:

1. The divergence produces a scalar field as the dot product is involved in the operation. The result does not have direction associated with it.

$$2. \nabla \cdot (\vec{A} + \vec{B}) = \nabla \cdot \vec{A} + \nabla \cdot \vec{B} \quad \text{Distributive}$$

MAXWELL'S FIRST EQUATION:

$$\text{div } \vec{D} = \nabla \cdot \vec{D} = \rho_v$$

This above equation is called Maxwell's first equation applied to electrostatics. This is also called the point form of Gauss's law (or) Gauss's law in differential form.

Problems

(1) Given, $\vec{A} = 2xy \vec{a}_x + z \vec{a}_y + yz^2 \vec{a}_z$

Find $\nabla \cdot \vec{A}$ at $P(2, -1, 3)$.

$$\begin{aligned} \nabla \cdot \vec{A} &= \text{div } \vec{A} = \frac{\partial A_x}{\partial x} + \frac{\partial A_y}{\partial y} + \frac{\partial A_z}{\partial z} \\ &= \frac{\partial}{\partial x} [2xy] + \frac{\partial}{\partial y} (z) + \frac{\partial}{\partial z} (yz^2) \\ &= 2y + 0 + 2zy \end{aligned}$$

At $P(2, -1, 3) \Rightarrow x=2, y=-1, z=3$.

$$\begin{aligned} \nabla \cdot \vec{A} &= 2(-1) + 0 + 2(3)(-1) \\ &= -2 - 6 = -8 \end{aligned}$$

$\Rightarrow \boxed{\nabla \cdot \vec{A} = -8}$

(2) Find the divergence of \vec{A} at $P(5, \pi/2, 1)$ where

$\vec{A} = r^2 z \sin \phi \vec{a}_r + 3r^2 z \cos \phi \vec{a}_\phi$ [in cylindrical system]

$$\text{div } \vec{A} = \frac{1}{r} \frac{\partial}{\partial r} (r A_r) + \frac{1}{r} \frac{\partial}{\partial \phi} A_\phi + \frac{\partial A_z}{\partial z}$$

$A_r = r^2 z \sin \phi \quad A_\phi = 3r^2 z \cos \phi \quad A_z = 0$

$$\text{div } \vec{A} = \frac{1}{r} \frac{\partial}{\partial r} [r^2 z \sin \phi] + \frac{1}{r} \frac{\partial}{\partial \phi} [3r^2 z \cos \phi]$$

~~$= \frac{1}{r} \frac{\partial}{\partial r} [r^2 z \sin \phi] + \frac{1}{r} \frac{\partial}{\partial \phi} [3r^2 z \cos \phi]$~~

$$= \frac{1}{r} \frac{\partial}{\partial r} [r^2 z \sin \phi] + \frac{1}{r} \cdot 3r^2 z \frac{\partial}{\partial \phi} [\cos \phi]$$

$$= \frac{1}{r} \cdot z \sin \phi [2r] + 3z^2 [-\sin \phi]$$

$$= 2z \sin \phi + 3z^2 [-\sin \phi]$$

$$\text{div } \vec{A} = 2z \sin \phi - 3z^2 \sin \phi$$

at point $P(5, \pi/2, 1) \quad r=5, \phi=\pi/2, z=1$

$$\text{div } \vec{A} = 2 \times 1 \times \sin \frac{\pi}{2} - 3 \times 1 \times \sin \frac{\pi}{2}$$

$$= 2 - 3$$

$$\boxed{\text{div } \vec{A} \Big|_P = -1}$$

DIVERGENCE THEOREM:

Gauss's law in integral form.

$$Q = \oint_S \vec{D} \cdot d\vec{s} \quad \dots \textcircled{1}$$

while charge enclosed in a volume is given by

$$Q = \int_V \rho_v \, dv \quad \dots \textcircled{2}$$

Gauss's law in Point form

$$\nabla \cdot \vec{D} = \rho_v \quad \dots \textcircled{3}$$

Sub $\textcircled{3}$ in $\textcircled{2}$ we get

$$Q = \int_V (\nabla \cdot \vec{D}) \, dv \quad \dots \textcircled{4}$$

Evaluating $\textcircled{1}$ & $\textcircled{4}$ we get

$$\boxed{\oint_S \vec{D} \cdot d\vec{s} = \int_V (\nabla \cdot \vec{D}) \, dv} \quad \dots \textcircled{5}$$

Euation $\textcircled{5}$ is called DIVERGENCE THEOREM. It is also called as GAUSS-OSTROGRADSKY THEOREM.

The integral of the normal component of any vector field over a closed surface is equal to the integral of the divergence of this vector field throughout the volume enclosed by that closed surface.

The theorem can be applied to any vector but partial derivatives of that vector field must exist.

With the help of the divergence theorem, the surface integral can be converted into a volume integral, provided that the closed surface encloses certain volume.

PROOF OF DIVERGENCE THEOREM:

According to divergence theorem, the surface integral is converted into a volume integral, provided that closed surface encloses certain volume.

Let the closed surface encloses certain volume V . Subdivide this volume V into a large number of subsections called cells.

Let the vector field associated with surface S is \vec{D} . Then if i^{th} cell has the volume ΔV_i and is bounded by the surface S_i , then we can write,

$$\oint_S \vec{D} \cdot d\vec{s} = \sum_i \oint_{S_i} \vec{D} \cdot d\vec{s} = \sum_i \frac{\oint_{S_i} \vec{D} \cdot d\vec{s}}{\Delta V_i} \cdot \Delta V_i \quad \text{--- (1)}$$

The cells are adjacent to each other hence the outward flux to one cell is inward to its neighbouring cells. Thus on every interior surface b/w the cells, there is cancellation of surface integrals and hence the sum of the surface integrals over surfaces S_i 's is equal to the total surface integrals over the entire surface S .

$$\therefore \oint_S \vec{D} \cdot d\vec{s} = \frac{\oint_S \vec{D} \cdot d\vec{s}}{\Delta V} \cdot \Delta V \quad \text{--- (2)}$$

Taking $\lim \Delta V$ tends to zero of RHS of (2) i.e. the volume shrinks about a point, the RHS of (2) gives divergence of \vec{D} ,

According to the definition of divergence,

$$\lim_{\Delta V \rightarrow 0} \frac{\oint_S \vec{D} \cdot d\vec{s}}{\Delta V} = \text{div } \vec{D} = \nabla \cdot \vec{D} \quad \text{--- (3)}$$

sub (3) in (2) we get

$$\oint_S \vec{D} \cdot d\vec{s} = [\nabla \cdot \vec{D}] \Delta V. \quad \text{--- (4)}$$

For considering entire volume, integrate RHS over the entire volume V , enclosed by the surface

$$\therefore \oint_S \vec{D} \cdot d\vec{s} = \int_V [\nabla \cdot \vec{D}] dv. \quad \text{--- (5)}$$

Equation (5) is the statement of the divergence theorem and hence divergence theorem is proved.

Problems

(i) Given that $\vec{A} = 30e^{-y}\vec{a}_1 - 2z\vec{a}_2$ in the cylindrical co-ordinates. Evaluate both sides of the divergence theorem for the volume enclosed by $r=2$, $z=0$ and $z=5$.

SOLUTION:-

The divergence theorem states that

$$\oint_S \vec{A} \cdot d\vec{s} = \int_V (\nabla \cdot \vec{A}) dv.$$

$$\text{Now } \oint_S \vec{A} \cdot d\vec{s} = \left[\oint_{\text{side}} + \oint_{\text{top}} + \oint_{\text{bottom}} \right] \vec{A} \cdot d\vec{s}.$$

consider $d\vec{s}$ normal to \vec{a}_1 direction which is for the side surface.

$$d\vec{s} = r d\phi dz \vec{a}_1$$

$$\vec{A} \cdot d\vec{s} = [30e^{-r}\vec{a}_r - 2z\vec{a}_z] \cdot r d\phi dz \vec{a}_r$$

$$= 30e^{-r} r d\phi dz (\vec{a}_r \cdot \vec{a}_r) - 2z r d\phi dz [\cancel{\vec{a}_z} \cdot \vec{a}_r]$$

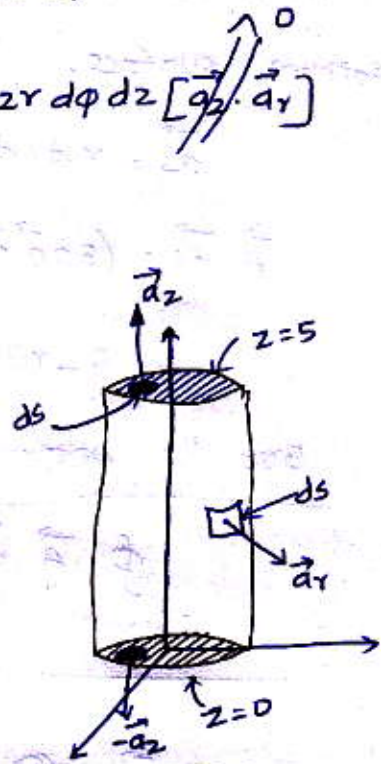
$$\vec{A} \cdot d\vec{s} = 30e^{-r} r d\phi dz$$

$$\oint_{\text{side}} \vec{A} \cdot d\vec{s} = \int_{\phi=0}^{2\pi} \int_{z=0}^5 30e^{-r} r d\phi dz$$

$$\Rightarrow 30e^{-r} [\phi]_0^{2\pi} [z]_0^5$$

$$\Rightarrow 30e^{-r} (2\pi)(5)$$

$$\Rightarrow 300\pi e^{-r}$$



Given that $r=2$

$$\therefore \oint_{\text{side}} \vec{A} \cdot d\vec{s} = 300\pi e^{-2} \cdot 2 = 600\pi e^{-2} = 255.1 \quad \text{--- (1)}$$

The $d\vec{s}$ on top has direction \vec{a}_z , hence for top surface

$$d\vec{s} = r dr d\phi \vec{a}_z$$

$$\vec{A} \cdot d\vec{s} = [30e^{-r}\vec{a}_r - 2z\vec{a}_z] \cdot r dr d\phi \vec{a}_z$$

$$= 30e^{-r} r dr d\phi [\cancel{\vec{a}_r} \cdot \vec{a}_z] - 2z r dr d\phi [\vec{a}_z \cdot \vec{a}_z]$$

$$= -2z r dr d\phi$$

$$\oint_{\text{top}} \vec{A} \cdot d\vec{s} = \oint_{\text{top}} -2z r dr d\phi \Rightarrow -2z \int_{\phi=0}^{2\pi} \int_{r=0}^2 r dr d\phi$$

$$\Rightarrow -2z [\phi]_0^{2\pi} \left[\frac{r^2}{2} \right]_0^2$$

$$= -2z (2\pi) \left(\frac{4}{2} \right)$$

$$\oint_{\text{top}} \vec{A} \cdot d\vec{s} = -8\pi z$$

w.k.t $z=5$

(25)

$$\therefore \oint_{\text{top}} \vec{A} \cdot d\vec{s} = -40\pi \quad \text{--- (2)}$$

while \vec{ds} for bottom has direction $(-\vec{a}_z)$ hence for bottom surface,

$$\vec{ds} = r dr d\phi (-\vec{a}_z)$$

$$\vec{A} \cdot \vec{ds} = (30e^{-r}\vec{a}_r - 2z\vec{a}_z) \cdot (r dr d\phi (-\vec{a}_z))$$

$$= -2z r dr d\phi$$

But for bottom surface $z = 0$

$$\oint_{\text{bottom}} \vec{A} \cdot \vec{ds} = 0 \quad \dots \textcircled{3}$$

$$\therefore \oint \vec{A} \cdot \vec{ds} = \textcircled{1} + \textcircled{2} + \textcircled{3}$$

$$= 255.1 - 40\pi$$

$$\oint \vec{A} \cdot \vec{ds} = 129.4363 = \text{LHS of divergence theorem.}$$

RHS of divergence theorem is $\int_V (\nabla \cdot \vec{A}) dv$

$$\nabla \cdot \vec{A} = \frac{1}{r} \frac{\partial}{\partial r} (r A_r) + \frac{1}{r} \frac{\partial A_\phi}{\partial \phi} + \frac{\partial A_z}{\partial z}$$

$$A_r = 30e^{-r} \quad A_\phi = 0 \quad A_z = -2z$$

$$\therefore \nabla \cdot \vec{A} = \frac{1}{r} \frac{\partial}{\partial r} (r 30e^{-r}) + 0 + \frac{\partial}{\partial z} (-2z)$$

$$= \frac{1}{r} [30r \{-e^{-r}\} + 30e^{-r}(1)] - 2$$

$$= -30e^{-r} + \frac{30}{r}e^{-r} - 2$$

In ~~Cartesian~~ cylindrical system $dv = r d\phi dr dz$

$$\int_V (\nabla \cdot \vec{A}) dv = \int_{z=0}^5 \int_{\phi=0}^{2\pi} \int_{r=0}^2 (-30e^{-r} + \frac{30}{r}e^{-r} - 2) r dr d\phi dz$$

$$= \int_{r=0}^2 \int_{\phi=0}^{2\pi} \int_{z=0}^5 (-30e^{-r} \cdot r + 30e^{-r} - 2r) dr d\phi dz$$

$$= \int_{r=0}^2 [30e^{-r} \cdot r + 30e^{-r} - 2r] dr * \int_{\phi=0}^{2\pi} d\phi * \int_{z=0}^5 dz$$

$$= -30 \int_{r=0}^2 r e^{-r} dr + 30 \int_{r=0}^2 e^{-r} dr - 2 \int_{r=0}^2 r dr * [\phi]_0^{2\pi} * [z]_0^5$$

$$u=r \quad dv=e^{-r}$$

$$= \left\{ -30 \left[r \left[\frac{e^{-r}}{-1} \right] - \int e^{-r} dr \right] + 30 \left[\frac{e^{-r}}{-1} \right] - 2 \left[\frac{r^2}{2} \right] \right\} \times 2\pi \times 5$$

[$\int u dv = uv - \int v du$]

$$= \left\{ \left[-30 \frac{r e^{-r}}{-1} - 30 \frac{e^{-r}}{-1} \right] - 30 [e^{-r}]_0^2 - [r^2]_0^2 \right\} 10\pi$$

$$= [30 r e^{-r} + 30 e^{-r} - 30 e^{-r} - r^2]_0^2 10\pi$$

$$= [60 e^{-2} - 2^2] \times 10\pi = 129.437$$

$\int_V (\nabla \cdot \vec{A}) dv = 129.437$

= RHS of divergence theorem.

ELECTRIC POTENTIAL:

Consider an Electric field due to a Positive charge Q . If a unit test Positive charge Q_t is placed at any point in this field, it experiences a repulsive force and tends to move in the direction of force.

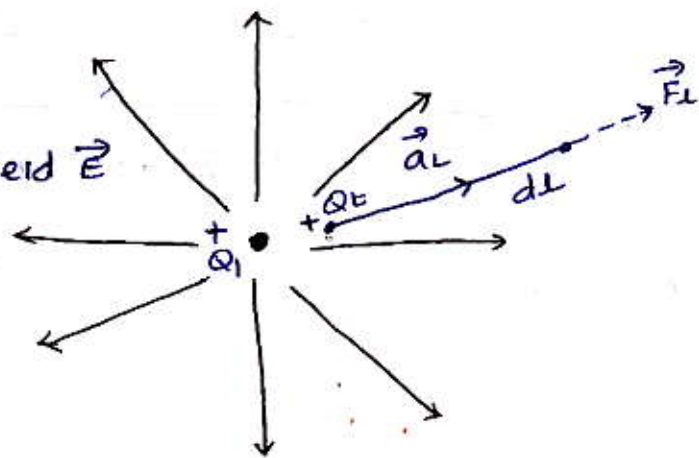
But if a Positive test charge Q_t is to be moved towards the Positive base charge Q then it is required to be moved against the electric field of charge Q i.e. against the repulsive force exerted by charge Q on the test charge Q_t .

While doing so, an external source has to do work to move the test charge Q_t against the electric field. This work done becomes the potential energy of the test charge Q_t , at the point at which it is moved.

Consider a Positive charge Q_1 and its electric field \vec{E} . If a Positive test charge Q_t is placed in this field, it will move due to force of repulsion. Let the movement of the charge Q_t is dL . The direction in which the movement has taken place is denoted by unit vector \vec{a}_L , in the direction of dL . This is shown in figure

According to Coulomb's law the force exerted by the field \vec{E} is given by,

$$\vec{F} = Q_t \vec{E} \text{ N.}$$

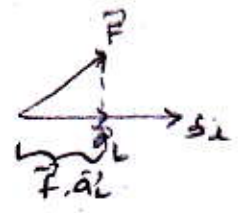


But the component of this force exerted by the field in the direction of dL , is responsible to move the charge q_t , thro' the distance dL .

By W.K.T the component of a vector in the direction of the unit vector is the dot product of the vector with that unit vector. Thus the component of \vec{F} in the direction of unit vector \vec{a}_L is given by,

$$\vec{F}_L = \vec{F} \cdot \vec{a}_L \quad \text{sub (1) in (2)}$$

$$\vec{F}_L = q_t \vec{E} \cdot \vec{a}_L \quad \text{N} \quad \text{--- (2)}$$



This is the force responsible to move the charge q_t thro' the distance dL , in the direction of the field.

To keep the charge in equilibrium, it is necessary to apply the force which is equal and opposite to the force exerted by the field in the direction dL .

$$\therefore \vec{F}_{\text{applied}} = -\vec{F}_L = -q_t \vec{E} \cdot \vec{a}_L \quad \text{N} \quad \text{--- (3)}$$

In this case, work is said to be done.

Mathematically the differential work done by an external source in moving the charge q_t , through a distance dL , against the direction of field \vec{E} is given by,

$$dW = \vec{F}_{\text{applied}} \times dL \quad \text{[work done = force} \times \text{distance]}$$

$$= -q_t \vec{E} \cdot \vec{a}_L dL$$

But $dL \vec{a}_L = d\vec{L} = \text{distance vector}$

$$dW = -q_t \vec{E} \cdot d\vec{L} \quad \text{J}$$

Scalar quantity

Thus if a charge Q is moved from initial position to final position, against the direction of electric field \vec{E} then the total work done is obtained by integrating the differential work done over the distance from initial position to the final position.

$$\therefore W = \int_{\text{Initial}}^{\text{Final}} dW = \int_{\text{Initial}}^{\text{Final}} -Q \vec{E} \cdot d\vec{L}$$

$$W = -Q \int_{\text{Initial}}^{\text{Final}} \vec{E} \cdot d\vec{L} \text{ J.}$$

This work done is measured in Joules.

THE LINE INTEGRAL:

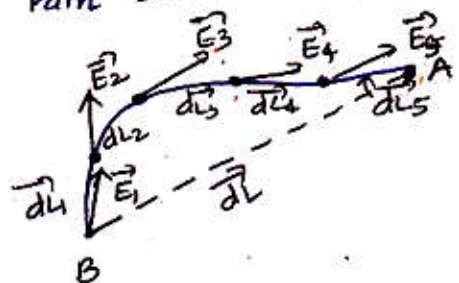
Consider that the charge is moved from initial position B to the final position A, against the electric field \vec{E} then the work done is given by,

$$W = -Q \int_B^A \vec{E} \cdot d\vec{L}$$

This is called the line integral, where $\vec{E} \cdot d\vec{L}$ is the component of \vec{E} along the direction $d\vec{L}$.

The line integral is basically a summation and accurate result is obtained when the number of segments becomes infinite.

Consider an uniform electric field \vec{E} . The charge is moved from B to A along the path shown in figure.



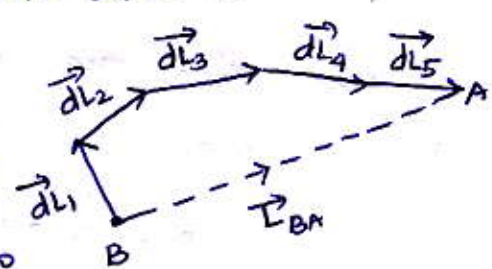
The path B to A is divided into number of small segments.

The various distance vectors along the segments chosen are $\vec{dl}_1, \vec{dl}_2, \vec{dl}_3, \vec{dl}_4$ and \vec{dl}_5 while Electric field in these directions is $\vec{E}_1, \vec{E}_2, \vec{E}_3, \vec{E}_4$ & \vec{E}_5 . Hence the line integral from B to A can be expressed as summation of dot products.

$$W = -Q [\vec{E}_1 \cdot \vec{dl}_1 + \vec{E}_2 \cdot \vec{dl}_2 + \dots + \vec{E}_5 \cdot \vec{dl}_5] \quad \text{--- (1)}$$

But the electric field is uniform and equal in all the directions.

$$\therefore \vec{E}_1 = \vec{E}_2 = \vec{E}_3 = \vec{E}_4 = \vec{E}_5 = \vec{E} \quad \text{--- (2)}$$



sub (2) in (1) leads to

$$\therefore W = -QE [\vec{dl}_1 + \vec{dl}_2 + \vec{dl}_3 + \dots + \vec{dl}_5] \quad \text{--- (3)}$$

Now $\vec{dl}_1 + \vec{dl}_2 + \dots + \vec{dl}_5$ is vector addition.

$$\text{If } \vec{dl}_1 + \vec{dl}_2 + \dots + \vec{dl}_5 = \vec{L}_{BA} \text{ then}$$

$$W = -QE \cdot \vec{L}_{BA}$$

Thus it can be seen that vector sum of small segments chosen along any path, a curve or a straight line remains same as \vec{L}_{BA} and it depends on initial and final point only.

Thus, the work done in moving a charge from one location B to another A, in a static, uniform or nonuniform electric field \vec{E} is independent of the path selected.

$$\begin{cases} \vec{dl} = dx \vec{a}_x + dy \vec{a}_y + dz \vec{a}_z & \text{[Cartesian system]} \\ \vec{dl} = dr \vec{a}_r + r d\phi \vec{a}_\phi + dz \vec{a}_z & \text{[Cylindrical]} \\ \vec{dl} = dr \vec{a}_r + r d\theta \vec{a}_\theta + r \sin\theta d\phi \vec{a}_\phi & \text{[Spherical]} \end{cases}$$

POTENTIAL DIFFERENCE:

The work done in moving a point charge Q from point B to A in the electric field \vec{E} is given by,

$$W = -Q \int_B^A \vec{E} \cdot d\vec{L} \quad \dots \text{--- (1)}$$

If the charge Q is selected as unit test charge then from the above equation we get the work done in moving unit charge from B to A in the field \vec{E} .

This work done in moving unit charge from point B to A in the field \vec{E} is called Potential difference, b/w the points B to A. It is denoted as V

$$\therefore V = - \int_B^A \vec{E} \cdot d\vec{L} \quad \dots \text{--- (2)}$$

Thus work done per unit charge in moving unit charge from B to A in the field \vec{E} is called Potential difference b/w the points B and A.

$$\underline{V_{AB}} = - \int_B^A \vec{E} \cdot d\vec{L}$$

Potential difference is work done per unit charge $\left(\frac{J}{C}\right)$ unit is

one volt potential difference is one Joule of work done in moving unit charge from one point to other in the field \vec{E}

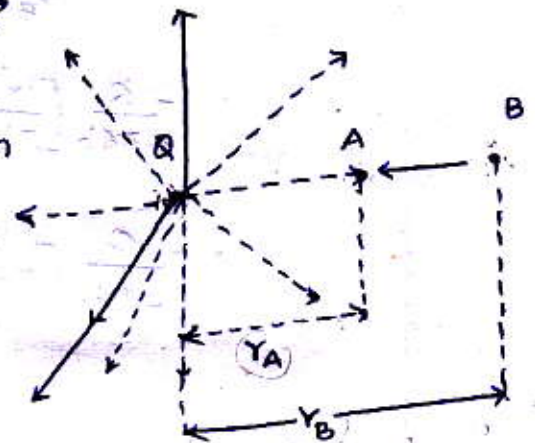
$$\therefore 1 \text{ Volt} = \frac{1 \text{ Joule}}{1 \text{ Coulomb}}$$

POTENTIAL DUE TO POINT CHARGE:

Consider a point charge, located at the origin of a spherical co-ordinate system, producing \vec{E} radially in all the directions as shown in figure.

Assuming free space, the field \vec{E} due to a point charge Q at a point having radial distance r from the origin is given by

$$\vec{E} = \frac{Q}{4\pi\epsilon_0 r^2} \vec{a}_r \quad \text{--- (1)}$$



Consider a unit charge which is placed at a point B which is at a radial distance of r_B from the origin. It is moved against the direction of \vec{E} from point B to point A. The point A is at a radial distance of r_A from the origin.

The differential length in spherical system is

$$d\vec{L} = dr \vec{a}_r + r d\theta \vec{a}_\theta + r \sin\theta d\phi \vec{a}_\phi$$

Hence potential difference V_{AB} between points A and B is given by

$$V_{AB} = - \int_B^A \vec{E} \cdot d\vec{L}$$

Here $B = r_B$
 $A = r_A$

$$\therefore V_{AB} = - \int_{r_B}^{r_A} \vec{E} \cdot d\vec{L}$$

$$= - \int_{r_B}^{r_A} \left(\frac{Q}{4\pi\epsilon_0 r^2} \vec{a}_r \right) \cdot (dr \vec{a}_r + r d\theta \vec{a}_\theta + r \sin\theta d\phi \vec{a}_\phi)$$

$$V_{AB} = - \int_{r_B}^{r_A} \frac{Q}{4\pi\epsilon_0 r^2} dr$$

$$= - \frac{Q}{4\pi\epsilon_0} \int_{r_B}^{r_A} r^{-2} dr$$

$$= - \frac{Q}{4\pi\epsilon_0} \left[\frac{r^{-1}}{-1} \right]_{r_B}^{r_A}$$

$$= - \frac{Q}{4\pi\epsilon_0} \left[-\frac{1}{r} \right]_{r_B}^{r_A} \Rightarrow - \frac{Q}{4\pi\epsilon_0} \left[-\frac{1}{r_A} - \left(-\frac{1}{r_B} \right) \right]$$

$$\Rightarrow - \frac{Q}{4\pi\epsilon_0} \left[-\frac{1}{r_A} + \frac{1}{r_B} \right]$$

$$V_{AB} = + \frac{Q}{4\pi\epsilon_0} \left[\frac{1}{r_A} - \frac{1}{r_B} \right] V$$

When $r_B > r_A$, $\frac{1}{r_B} < \frac{1}{r_A}$ and V_{AB} is +ve. This indicates work is done by external source in moving unit charge from B to A.

CONCEPT OF ABSOLUTE POTENTIAL:

Instead of potential difference, it is more convenient to express absolute potentials at various points in the field. Such absolute potentials are measured w.r. to a specified reference position. Such reference position is assumed to be at zero potential.

For practical circuits, zero reference point is selected as ground.

Consider potential difference V_{AB} due to movement of unit charge from B to A. in to a field of a point charge Q. It is given by

$$V_{AB} = \frac{Q}{4\pi\epsilon_0} \left[\frac{1}{r_A} - \frac{1}{r_B} \right] \text{--- (A)}$$

Now let the charge is moved from infinity to the Point A i.e. $r_B = \infty$. Hence $\frac{1}{r_B} = \frac{1}{\infty} = 0$.

$$\therefore V_{AB} = \frac{Q}{4\pi\epsilon_0} \left[\frac{1}{r_A} - \frac{1}{\infty} \right] = \frac{Q}{4\pi\epsilon_0 r_A} \quad \text{--- (B)}$$

The quantity represented in eqn (B) is called potential of Point A denoted as V_A .

$$\boxed{V_A = \frac{Q}{4\pi\epsilon_0 r_A}} \rightarrow \text{This is also called absolute potential of Point A.}$$

Similarly absolute potential of Point B can be defined as

$$\boxed{V_B = \frac{Q}{4\pi\epsilon_0 r_B}}$$

This is work done in moving unit charge from ∞ at Point B.

Hence the potential difference can be expressed as the difference b/w the absolute potentials of the two points.

$$\boxed{\therefore V_{AB} = V_A - V_B}$$

POTENTIAL DUE TO POINT CHARGE NOT AT ORIGIN:

If the point charge Q is not located at the origin of a spherical system then obtain the position vector r' of the point Q where Q is located.

Then absolute potential at a Point A located at a distance r from the origin is given by,

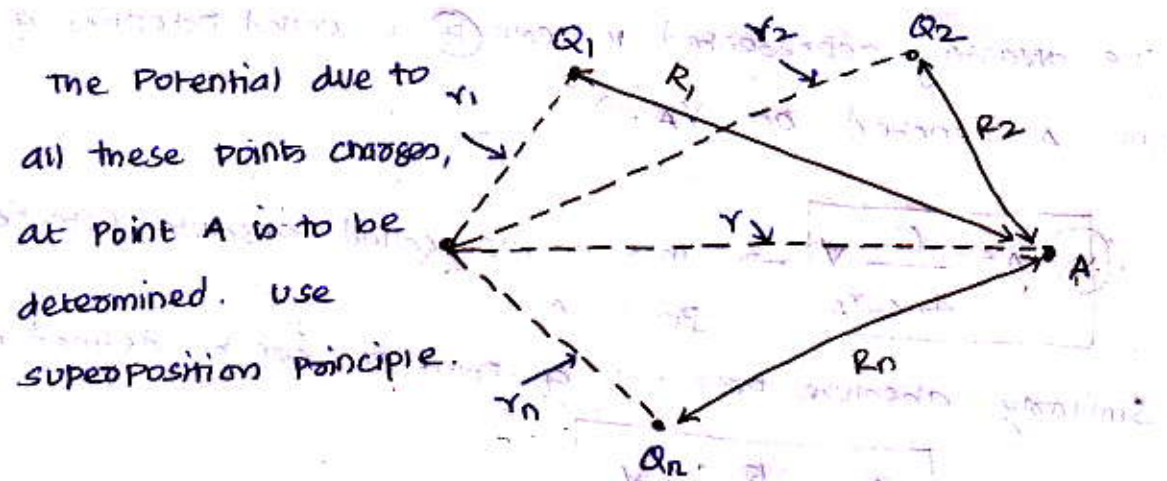
$$\begin{aligned} V(r) = V_A &= \frac{Q}{4\pi\epsilon_0 |r-r'|} \\ &= \frac{Q}{4\pi\epsilon_0 R_A} \end{aligned}$$



where R_A is the distance b/w Point at which potential is to be calculated and the location of the charge.

POTENTIAL DUE TO SEVERAL POINT CHARGES:

Consider the various point charges Q_1, Q_2, \dots, Q_n located at the distance of r_1, r_2, \dots, r_n from the origin as shown in the figure.



The Potential due to all these points charges, at point A is to be determined. Use superposition principle.

Consider the point charge Q_1

The Potential V_{A1} due to Q_1 is given by

$$V_{A1} = \frac{Q_1}{4\pi\epsilon_0 |r - r_1|} = \frac{Q_1}{4\pi\epsilon_0 R_1} V$$

where $R_1 = |r - r_1|$ = Distance b/w point A and position of Q_1 .

The Potential V_{A2} due to Q_2 is given by

$$V_{A2} = \frac{Q_2}{4\pi\epsilon_0 |r - r_2|} = \frac{Q_2}{4\pi\epsilon_0 R_2} V$$

Thus Potential V_{An} due to Q_n is given by

$$V_{An} = \frac{Q_n}{4\pi\epsilon_0 |r - r_n|} = \frac{Q_n}{4\pi\epsilon_0 R_n} V$$

The net Potential at point A is the algebraic sum of the Potentials at A due to individual point charges is

$$V_A = V_{A1} + V_{A2} + \dots + V_{An} = \frac{Q_1}{4\pi\epsilon_0 R_1} + \frac{Q_2}{4\pi\epsilon_0 R_2} + \dots + \frac{Q_n}{4\pi\epsilon_0 R_n}$$

$$= \sum_{m=1}^n \frac{Q_m}{4\pi\epsilon_0 |r - r_m|} = \sum_{m=1}^n \frac{Q_m}{4\pi\epsilon_0 R_m} V$$

Problems:

1. A point charge $Q = 0.4 \text{ nC}$ is located at the origin. obtain the absolute potential of $A(2, 2, 3)$.

SOLUTION:

The potential of A due to point charge Q at the origin is given by

$$V_A = \frac{Q}{4\pi\epsilon_0 R_A} \quad \text{and } A(2, 2, 3), \quad Q \text{ at } (0, 0, 0)$$

$$\therefore R_A = \sqrt{(2-0)^2 + (2-0)^2 + (3-0)^2} = \sqrt{17}$$

$$V_A = \frac{0.4 \times 10^{-9}}{4\pi \times 8.854 \times 10^{-12} \times \sqrt{17}} = 0.8179 \text{ V.}$$

2. If the same charge $Q = 0.4 \text{ nC}$ in the above example is located at $(2, 3, 3)$ then obtain the absolute potential of point $A(2, 2, 3)$.

SOLUTION:

Now Q is located at $(2, 3, 3)$.

$$\therefore V_A = \frac{Q}{4\pi\epsilon_0 |r-r'|} = \frac{Q}{4\pi\epsilon_0 R_A}$$

$$R_A = \sqrt{(2-2)^2 + (2-3)^2 + (3-3)^2} = 1$$

$$\therefore V_A = \frac{0.4 \times 10^{-9}}{4\pi \times 8.854 \times 10^{-12}} = 3.595 \text{ V.}$$

EQUIPOTENTIAL SURFACES:

In an electric field, there are many points at which the electric potential is same. This is because, the potential is a scalar quantity which depends on the distance b/w the point at which potential is to be obtained and location of the charge.

All such points are at the same electric potential.

If a surface is imagined, joining all such points which are at the (same) potential, then such a surface is called Equipotential surface.

CONSERVATIVE FIELD

The work done in moving a test charge around any closed path in a static field \vec{E} is zero. This is because starting and terminating point is same for a closed path. Hence upper and lower limit of integration becomes same. Hence the work done becomes zero. such an integral over a closed path is denoted

$$\oint_{\text{closed path}} \vec{E} \cdot d\vec{l} = 0 \rightarrow \text{conservative field (or) lamellar field}$$

POTENTIAL GRADIENT:

$$\vec{E} = -\nabla V$$

$$E_x = -\frac{\partial V}{\partial x}, E_y = -\frac{\partial V}{\partial y}, E_z = -\frac{\partial V}{\partial z}$$

... of which ... the electric potential ...

Hence an inverse relation namely the change of potential ΔV , along the elementary length ΔL must be related to \vec{E} , as $\Delta L \rightarrow 0$

(*) [The rate of change of potential with respect to the distance is called Potential gradient]

$$\frac{dV}{dL} = \lim_{\Delta L \rightarrow 0} \frac{\Delta V}{\Delta L} = \text{Potential gradient}$$

RELATIONSHIP BETWEEN \vec{E} and V

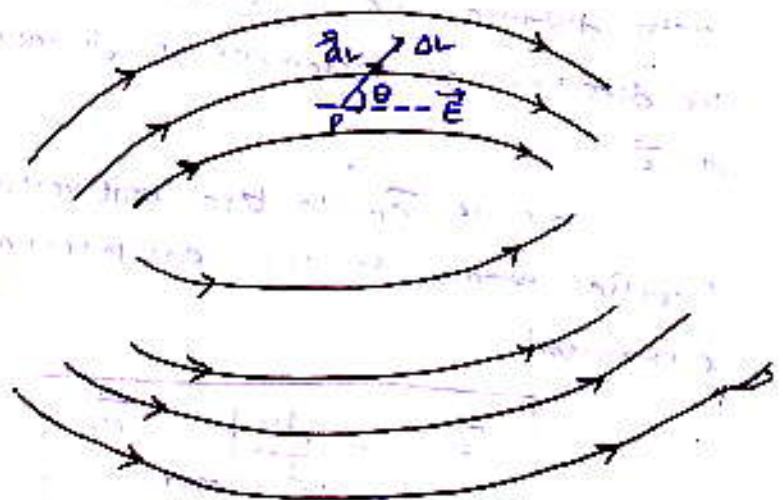
consider \vec{E} due to a particular charge distribution in space. The electric field \vec{E} and potential V is changing from point to point in space.

Consider a vector incremental length $\vec{\Delta L}$ making an angle θ w.r. to the direction \vec{E} as shown in figure.

To find incremental potential we use

$$\Delta V = -\vec{E} \cdot \vec{\Delta L} \quad \text{--- (1)}$$

$$\vec{\Delta L} = \Delta L \vec{a}_L \quad \text{--- (2)}$$



(1) \Rightarrow

$$\Delta V = -\vec{E} \cdot \vec{\Delta L}$$

$$= -E \Delta L \cos \theta$$

$$\frac{\Delta V}{\Delta L} = -E \cos \theta$$

To find ΔV at a point, take $\lim_{\Delta L \rightarrow 0}$

$$\therefore \lim_{\Delta L \rightarrow 0} \frac{\Delta V}{\Delta L} = -E \cos \theta$$

$$\text{But } \lim_{\Delta L \rightarrow 0} \frac{\Delta V}{\Delta L} = \frac{dV}{dL} = \text{Potential Gradient}$$

$$\therefore \frac{dV}{dL} = -E \cos \theta$$

Hence the Potential gradient $\frac{dV}{dL}$ can be maximum only when $\cos \theta = -1$ i.e. $\theta = +180^\circ$.

This indicates that ΔL must be in the direction opposite to \vec{E} .

$$\left. \frac{dV}{dL} \right|_{\max} = E$$

Thus the above equations shows that,

1. Maximum value of Potential gradient gives the magnitude of the electric field intensity \vec{E} .
2. The maximum value of rate of change of Potential with distance, i.e. Potential gradient is possible only when the direction of increment in distance is opposite to the direction of \vec{E} .

Thus if \vec{a}_n is the unit vector in the direction of increasing Potential normal to the equipotential surface then \vec{E} can be expressed.

$$\vec{E} = - \left. \frac{dV}{dL} \right|_{\max} \vec{a}_n$$

As \vec{E} and Potential gradient are in opposite direction, above equation has -ve sign.

The maximum value of rate of change of Potential with distance (dV/dL) is called gradient of V .

mathematically

$$\text{Gradient of } V = \text{grad } V = \nabla V$$

$$\therefore \vec{E} = -\nabla V$$

The grad V in various co-ordinates are given as

1. Cartesian $\nabla V = \frac{\partial V}{\partial x} \vec{a}_x + \frac{\partial V}{\partial y} \vec{a}_y + \frac{\partial V}{\partial z} \vec{a}_z$

2. cylindrical $\nabla V = \frac{\partial V}{\partial r} \vec{a}_r + \frac{1}{r} \frac{\partial V}{\partial \phi} \vec{a}_\phi + \frac{\partial V}{\partial z} \vec{a}_z$

3. spherical $\nabla V = \frac{\partial V}{\partial r} \vec{a}_r + \frac{1}{r} \frac{\partial V}{\partial \theta} \vec{a}_\theta + \frac{1}{r \sin \theta} \frac{\partial V}{\partial \phi} \vec{a}_\phi$

AN ELECTRIC DIPOLE:

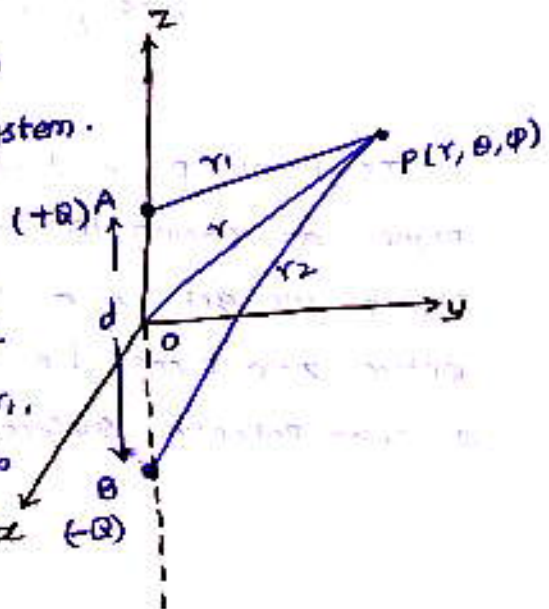
Two point charges of equal magnitude but opposite sign, separated by a very small distance gives rise to an ELECTRIC DIPOLE.

The field produced by such a dipole plays an important role in the engineering electromagnetics.

Consider an electric dipole as shown in the figure. The two point charges $+Q$ and $-Q$ are separated by a very small distance d .

Consider an ~~electric~~ point 'P' $P(r, \theta, \phi)$ in spherical co-ordinate system.

It is required to find \vec{E} due to an electric dipole at point P. Let 'O' be the midpoint of AB. The distance of point P from A is r_1 . While distance of point P from B is r_2 . The distance of point P from O is r .



The distance of separation of charges is d , it is very small compared to r_1 , r_2 and r .

The co-ordinates of A is $(0, 0, +\frac{d}{2})$ and B $(0, 0, -\frac{d}{2})$.

To find \vec{E} , we will find out the potential V at point P, due to an electric dipole. Then using $\vec{E} = -\nabla V$, we can find \vec{E} due to an electric dipole.

EXPRESSION OF \vec{E} DUE TO AN ELECTRIC DIPOLE:

In spherical co-ordinates, the potential at point P due to the charge $+Q$ is given by,

$$V_1 = \frac{+Q}{4\pi\epsilon_0 r_1} \quad \text{--- (1)}$$

The potential at P due to the charge $-Q$ is given by

$$V_2 = \frac{-Q}{4\pi\epsilon_0 r_2} \quad \text{--- (2)}$$

The total potential at point P is the algebraic sum of

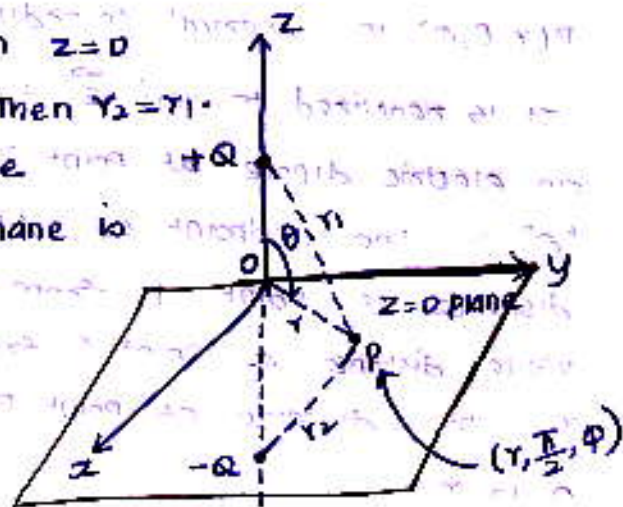
V_1 and V_2

$$\therefore V = V_1 + V_2$$

$$= \frac{+Q}{4\pi\epsilon_0 r_1} - \frac{Q}{4\pi\epsilon_0 r_2}$$

$$\therefore V = \frac{Q}{4\pi\epsilon_0} \left[\frac{1}{r_1} - \frac{1}{r_2} \right] = \frac{Q}{4\pi\epsilon_0} \left[\frac{r_2 - r_1}{r_1 r_2} \right] \quad \text{--- (3)}$$

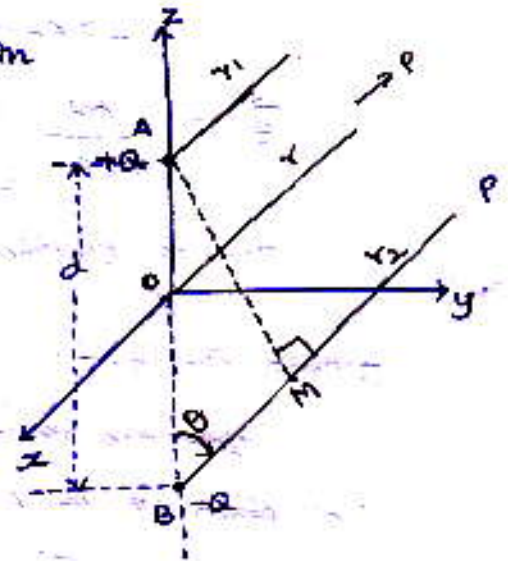
If point P is located in $z=0$ plane as shown in figure, then $r_2 = r_1$. Hence we get $V=0$. Thus the entire $z=0$ plane i.e. xy plane is a zero potential surface.



Now consider that P is located far away from the electric dipole. Thus r_1, r_2 and r can be assumed to be \parallel^{el} to each other, as shown in figure (b).

AM is drawn \perp from A on r_2 .

The angle made by r_1, r_2 and r with z axis is θ as all are \parallel .



$$\therefore BM = AB \cos \theta$$

$$= d \cos \theta \quad \text{--- (1)}$$

$$PB = PM + BM$$

and $PA = PM$ as AM is \perp

$$PB = r_2 \quad PA = r_1$$

$$\therefore BM = PB - PM$$

$$BM = r_2 - r_1 \quad \text{--- (2)}$$

$$(1) = (2) \quad \therefore r_2 - r_1 = d \cos \theta$$

As d is very small, $r_1 \approx r_2 \approx r$ hence $r_1 r_2 = r^2$ } --- (3)

Sub (3) in (2) we get

$$V = \frac{Q}{4\pi\epsilon_0} \left[\frac{d \cos \theta}{r^2} \right] \text{ Volts} \quad \text{--- (4)}$$

$$\text{Now } \vec{E} = -\nabla V = - \left[\frac{\partial V}{\partial r} \vec{a}_r + \frac{1}{r} \frac{\partial V}{\partial \theta} \vec{a}_\theta + \frac{1}{r \sin \theta} \frac{\partial V}{\partial \phi} \vec{a}_\phi \right]$$

$$\frac{\partial V}{\partial r} = \frac{\partial}{\partial r} \left[\frac{Qd \cos \theta}{4\pi\epsilon_0 r^2} \right] = \frac{Qd \cos \theta}{4\pi\epsilon_0} \frac{\partial}{\partial r} \left(\frac{1}{r^2} \right)$$

$$= \frac{Qd \cos \theta}{4\pi\epsilon_0} \frac{\partial}{\partial r} (r^{-2})$$

$$\frac{\partial V}{\partial r} = -\frac{2Qd \cos \theta}{4\pi\epsilon_0} r^{-3} \Rightarrow -\frac{2Qd \cos \theta}{4\pi\epsilon_0 r^3}$$

$$\frac{\partial V}{\partial \theta} = \frac{Qd}{4\pi\epsilon_0 r^2} [-\sin \theta] \quad \frac{\partial V}{\partial \phi} = 0$$

$$\therefore \vec{E} = - \left[\frac{-2Qd \cos \theta}{4\pi\epsilon_0 r^3} \vec{a}_r - \frac{Qd \sin \theta}{4\pi\epsilon_0 r^2} \vec{a}_\theta \right]$$

$$\vec{E} = \frac{2Qd \cos\theta}{4\pi\epsilon_0 r^3} \vec{a}_r + \frac{Qd \sin\theta}{4\pi\epsilon_0 r^3} \vec{a}_\theta$$

$$\vec{E} = \frac{Qd}{4\pi\epsilon_0 r^3} [2 \cos\theta \vec{a}_r + \sin\theta \vec{a}_\theta]$$

This is Electric field \vec{E} at Point P due to an electric dipole.

DIPOLE MOMENT:

Let the vector length directed from $-Q$ to $+Q$ i.e. from B to A is \vec{d}

$$\therefore \vec{d} = d \vec{a}_z \text{ --- (1)}$$

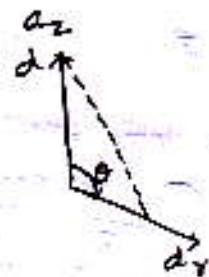
Its component along \vec{a}_r direction can be obtained as,

$$dr = \vec{d} \cdot \vec{a}_r \text{ --- (2)}$$

sub (1) in (2) we get

$$dr = d \vec{a}_z \cdot \vec{a}_r = d \cos\theta$$

$$\therefore \vec{d} = d \cos\theta \vec{a}_r$$



Then the product of $Q\vec{d}$ is called dipole moment and denoted as \vec{P}

$$\vec{P} = Q\vec{d}$$

The dipole moment is measured in Cm (Coulomb-meter).

$$\text{Now } \vec{P} \cdot \vec{a}_r = Q\vec{d} \cdot \vec{a}_r = Qd \cos\theta$$

Hence the expression of potential V can be expressed

$$V = \frac{Qd \cos\theta}{4\pi\epsilon_0 r^2} = \frac{\vec{P} \cdot \vec{a}_r}{4\pi\epsilon_0 r^2} \quad V$$

from Equation (A) of page no (43).

Note:

\vec{a}_r is the unit vector in the direction of distance vector joining the point at which moment exists and point at which V is to be obtained. $= \frac{\vec{r}}{r}$ \vec{r} = vector joining point of dipole moment to P.

Problem

A dipole having moment $\vec{p} = 3\vec{a}_x - 5\vec{a}_y + 10\vec{a}_z$ nCm is located at $Q(1, 2, -4)$ in free space. Find V at $P(2, 3, 4)$

Solution:

The potential V in terms of dipole moment is

$$V = \frac{\vec{p} \cdot \vec{a}_r}{4\pi\epsilon_0 r^2}$$

$Q(1, 2, -4)$ and $P(2, 3, 4)$

$$\begin{aligned}\vec{r} &= (2-1)\vec{a}_x + (3-2)\vec{a}_y + (4-(-4))\vec{a}_z \\ &= \vec{a}_x + \vec{a}_y + 8\vec{a}_z\end{aligned}$$

$$|\vec{r}| = \sqrt{1+1+64} = \sqrt{66}$$

$$\therefore \vec{a}_r = \frac{\vec{r}}{|\vec{r}|} = \frac{\vec{a}_x + \vec{a}_y + 8\vec{a}_z}{\sqrt{66}}$$

$$\vec{p} \cdot \vec{a}_r = (3\vec{a}_x - 5\vec{a}_y + 10\vec{a}_z) \cdot \left(\frac{\vec{a}_x + \vec{a}_y + 8\vec{a}_z}{\sqrt{66}} \right)$$

$$= \frac{3-5+80}{\sqrt{66}} = \frac{78}{\sqrt{66}} \times 10^{-9} \text{ as } \vec{p} \text{ is in nCm.}$$

$$V = \frac{\vec{p} \cdot \vec{a}_r}{4\pi\epsilon_0 r^2} = \frac{(78/\sqrt{66}) \times 10^{-9}}{4\pi \times 8.854 \times 10^{-12} \times (\sqrt{66})^2} = 1.3074 \text{ V.}$$