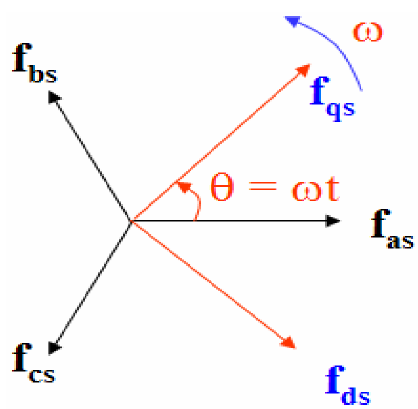
Arbitrary Reference Frame Theory

Synchronous and induction machine inductances are functions of the rotor speed, therefore the coefficients of the differential equations (voltage equations) which describe the behavior of these machines are time-varying. A change of variables can be used to reduce the complexity of machine differential equations, and represent these equations in another reference frame with constant coefficients. A change of variables which formulates a transformation of the 3-phase variables of stationary circuit elements to the arbitrary reference frame may be expressed as



$$\mathbf{f}_{qd0s} = \mathbf{K}_s \mathbf{f}_{abcs}$$

$$where, \quad (\mathbf{f}_{qd0s})^T = \begin{bmatrix} f_{qs} & f_{ds} & f_{0s} \end{bmatrix},$$

$$(\mathbf{f}_{abcs})^T = \begin{bmatrix} f_{as} & f_{bs} & f_{cs} \end{bmatrix},$$

$$\mathbf{K}_s = \frac{2}{3}\begin{bmatrix} \cos\theta & \cos(\theta - \frac{2\pi}{3}) & \cos(\theta + \frac{2\pi}{3}) \\ \sin\theta & \sin(\theta - \frac{2\pi}{3}) & \sin(\theta + \frac{2\pi}{3}) \\ \frac{1}{2} & \frac{1}{2} & \frac{1}{2} \end{bmatrix},$$

$$\theta = \int_0^t \omega(t)dt + \theta(0).$$

$$(\mathbf{K}_s)^{-1} = \begin{bmatrix} \cos\theta & \sin\theta & 1 \\ \cos(\theta - \frac{2\pi}{3}) & \sin(\theta - \frac{2\pi}{3}) & 1 \\ \cos(\theta + \frac{2\pi}{3}) & \sin(\theta + \frac{2\pi}{3}) & 1 \end{bmatrix}.$$

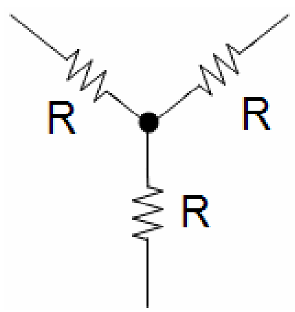
"f" can represent either voltage, current, or flux linkage."s" indicates the variables, parameters and transformation associated with stationary circuits."ω" represent the speed of reference frame.ω=0: Stationary reference frame.ω=ωe: synchronously rotating reference frame.ω=ωr: rotor reference frame (i.e., the reference frame is fixed on the rotor).f_{as}, f_{bs} and f_{cs} may be thought of as the direction of the magnetic axes of the stator windings.f_{qs} and f_{ds} can be considered as the direction of the magnetic axes of the "new" fictious windings located on qs and ds axis which are created by the change of variables. Power Equations corresponding to the transformations is given by

$$P_{abcs} = V_{as}i_{as} + V_{bs}i_{bs} + V_{cs}i_{cs}$$

$$P_{qd0s} = P_{abcs} = \frac{3}{2}\left(V_{qs}i_{qs} + V_{ds}i_{ds} + 2V_{0s}i_{0s}\right)$$

Stationary circuit variables transformed to the arbitrary reference frame

Resistive elements: For a 3-phase resistive circuit



$$V_{abcs} = \bar{r}_s i_{abcs} \qquad V_{qd0s} = \bar{r}_s i_{qd0s}$$

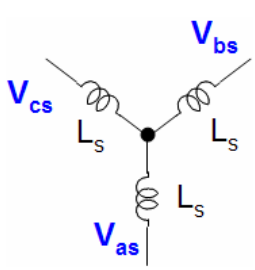
$$V_{abcs} = (\mathbf{K}_s)^{-1} V_{qd0s} \qquad , (\mathbf{K}_s)\bar{r}_s(\mathbf{K}_s)^{-1} = r_s$$

$$i_{abcs} = (\mathbf{K}_s)^{-1} i_{qd0s}$$

$$(\mathbf{K}_s)^{-1} V_{qd0s} = \bar{r}_s (\mathbf{K}_s)^{-1} i_{qd0s} \qquad \bar{r}_s = \begin{bmatrix} r_s & 0 & 0 \\ 0 & r_s & 0 \\ 0 & 0 & r_s \end{bmatrix}$$

$$V_{qd0s} = (\mathbf{K}_s)\bar{r}_s(\mathbf{K}_s)^{-1} i_{qd0s}$$

Inductive elements: For a 3-phase inductive circuit



$$V_{abcs} = p\lambda_{abcs},$$

$$where,\ p = \frac{d}{dt},$$

$$\lambda_{abcs} = \mathbf{L}_s i_{abcs} = \begin{bmatrix} L_s & 0 & 0 \\ 0 & L_s & 0 \\ 0 & 0 & L_s \end{bmatrix}\begin{bmatrix} i_{as} \\ i_{bs} \\ i_{cs} \end{bmatrix}$$

In terms of the substitute variables, we have

$$\mathbf{V}_{qd0s} = \mathbf{K}_s \cdot p\left[\mathbf{K}_s^{-1}\lambda_{qd0s}\right] = \mathbf{K}_s \cdot p\left[\mathbf{K}_s^{-1}\right]\lambda_{qd0s} + \mathbf{K}_s \cdot \left[\mathbf{K}_s^{-1}\right]p\lambda_{qd0s}$$

$$where,\ p\left[\mathbf{K}_s^{-1}\right] = \omega \cdot \begin{bmatrix} -\sin\theta & \cos\theta & 0 \\ -\sin(\theta - \frac{2\pi}{3}) & \cos(\theta - \frac{2\pi}{3}) & 0 \\ -\sin(\theta + \frac{2\pi}{3}) & \cos(\theta + \frac{2\pi}{3}) & 0 \end{bmatrix}$$

$$\mathbf{K}_s \cdot p\left[\mathbf{K}_s^{-1}\right] = \omega \cdot \begin{bmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$V_{qd0s} = \mathbf{K}_s p[(\mathbf{K}_s)^{-1}]\lambda_{qd0s} + \mathbf{K}_s(\mathbf{K}_s)^{-1} p\lambda_{qd0s}$$

$$V_{qd0s} = \quad \omega\lambda_{qds} \quad + \quad p\lambda_{qd0s}$$

$$where,\quad (\lambda_{qds})^T = \left[\lambda_{ds} \quad -\lambda_{qs} \quad 0\right]$$

Vector equation V_{qd0}s can be expressed as

$$V_{qs} = \omega\lambda_{ds} + p\lambda_{qs}$$
$$V_{ds} = -\omega\lambda_{qs} + p\lambda_{ds}$$
$$V_{0s} = p\lambda_{0s}$$

Where "$\omega\lambda_{ds}$" term and "$\omega\lambda_{qs}$" term are referred to as a "speed voltage" with the speed being the angular velocity of the arbitrary reference frame. When the reference frame is fixed in the stator, that is, the stationary reference frame (ω=0), the voltage equations for the three-phase circuit become the familiar time rate of change of flux linkage in abcs reference frame. For the three-phase circuit shown, Ls is a diagonal matrix, and

$$\lambda_{abcs} = \mathbf{L}_s i_{abcs}$$

$$\lambda_{qd0s} = \mathbf{K}_s\mathbf{L}_s\mathbf{K}_s^{-1} i_{qd0s} = \mathbf{L}_s i_{qd0s}$$

For the three-phase induction or synchronous machine, Ls matrix is expressed as

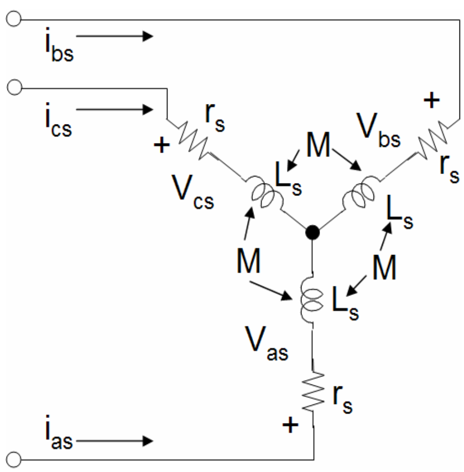
$$\mathbf{L}_s = \begin{bmatrix} L_{ls} + L_{ms} & -\frac{1}{2}L_{ms} & -\frac{1}{2}L_{ms} \\ -\frac{1}{2}L_{ms} & L_{ls} + L_{ms} & -\frac{1}{2}L_{ms} \\ -\frac{1}{2}L_{ms} & -\frac{1}{2}L_{ms} & L_{ls} + L_{ms} \end{bmatrix}$$

Where, L_{ls}: leakage inductance, L_{ms}: magnetizing inductance

$$\mathbf{K}_s \mathbf{L}_s (\mathbf{K}_s)^{-1} = \begin{bmatrix} \mathbf{L}_{ls} + \frac{3}{2}\mathbf{L}_{ms} & 0 & 0 \\ 0 & \mathbf{L}_{ls} + \frac{3}{2}\mathbf{L}_{ms} & 0 \\ 0 & 0 & \mathbf{L}_{ls} + \frac{3}{2}\mathbf{L}_{ms} \end{bmatrix}$$

Inductive elements: For a 3-phase inductive circuit with mutual inductance

Consider the stator windings of a symmetrical induction or round rotor synchronous machine shown below



$$\mathbf{r}_s = diag\begin{bmatrix} r_s & r_s & r_s \end{bmatrix}$$

$$\mathbf{L}_s = \begin{bmatrix} L_s & M & M \\ M & L_s & M \\ M & M & L_s \end{bmatrix}$$

$$L_s = L_{ls} + L_{ms}$$

$$M = -\frac{1}{2}L_{ms}$$

For each phase voltage, we write the following equations,

$$V_{as} = r_s i_{as} + p\lambda_{as}$$
$$V_{bs} = r_s i_{bs} + p\lambda_{bs},$$
$$V_{cs} = r_s i_{cs} + p\lambda_{cs}$$

$$V_{qd0s} = \mathbf{K}_s V_{abcs}$$
$$i_{qd0s} = \mathbf{K}_s i_{abcs}$$
$$\lambda_{qd0s} = \mathbf{K}_s \lambda_{abcs}$$
$$\lambda_{abcs} = \mathbf{L}_s i_{abcs}$$

In vector form,

$$V_{abcs} = \mathbf{r}_s i_{abcs} + p\lambda_{abcs},$$

Multiplying by $\mathbf{K}_s$

$$\mathbf{K}_s V_{abcs} = \mathbf{K}_s \mathbf{r}_s i_{abcs} + \mathbf{K}_s p\lambda_{abcs}$$

Replace i_{abcs} and λ_{abcs} using the transformation equations

$$\mathbf{K}_s V_{abcs} = \mathbf{K}_s \mathbf{r}_s (\mathbf{K}_s^{-1} i_{qd0s}) + \mathbf{K}_s p(\mathbf{K}_s^{-1}\lambda_{qd0s})$$

$$V_{qd0s} = \mathbf{r}_s i_{qd0s} + \overline{\omega}\overline{\lambda}_{qd0s}$$

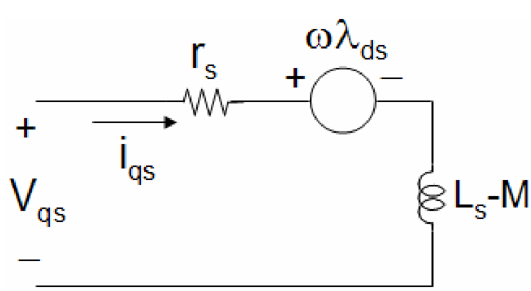
Or

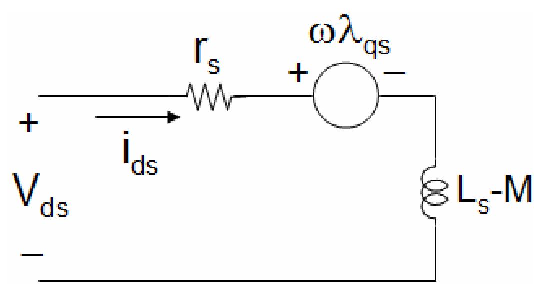
$$V_{qs} = r_s i_{qs} + \omega\lambda_{ds} + p\lambda_{qs}$$
$$V_{ds} = r_s i_{ds} - \omega\lambda_{qs} + p\lambda_{ds}$$
$$V_{0s} = r_s i_{0s} + p\lambda_{0s}$$

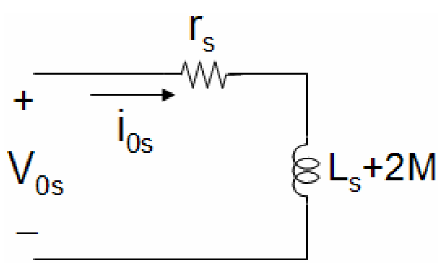
$$\text{where, } \overline{\omega} = \omega\begin{bmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$\lambda_{qs} = (L_s - M)i_{qs}$$
$$\lambda_{ds} = (L_s - M)i_{ds}$$
$$\lambda_{0s} = (L_s + 2M)i_{0s}$$

Our equivalent circuit in arbitrary reference frame can be represented as







ω=unspecified: stationary circuit variables referred to the arbitrary reference frame. The variables are referred to as f_{qd0s} or f_{qs}, f_{ds} and f_{0s} and transformation matrix is designated as K_s. ω=0: stationary circuit variables referred to the stationary reference frame. The variables are referred to as f^s_{qd0s} or f^s_{qs}, f^s_{ds} and f^s_{0s} and transformation matrix is designated as K^s_s. ω= ωr: stationary circuit variables referred to the reference frame fixed in the rotor. The variables are referred to as f^r_{qd0s} or f^r_{qs}, f^r_{ds} and

f^r_{0s} and transformation matrix is designated as K^r_s. ω= ωe: stationary circuit variables referred to the synchronously rotating reference frame. The variables are referred to as f^e_{qd0s} or f^e_{qs}, f^e_{ds} and f^e_{0s} and transformation matrix is designated as K^e_s.

Representation

f^{s}_{qd0s} → s: Stationary reference frame; qd0s: q-d axes of stator variables

f^{r}_{qd0s} → r: Reference frame fixed on the rotor with speed of ω_r; qd0s: q-d axes of stator variables $\theta_r = \int_0^t \omega_r(t)dt$

f^{e}_{qd0s} → e: Synchronously rotating reference frame; qd0s: q-d axes of stator variables $\theta_e = \int_0^t \omega_e(t)dt$

Transformation of a Balanced Set

Consider a 3-phase circuit which is excited by a balanced 3-phase voltage set. Assume the balanced set is a set of equal amplitude sinusoidal quantities which are displaced by 120°.

$$f_{as} = \sqrt{2} f_s \cos\theta_{ef}$$

$$f_{bs} = \sqrt{2} f_s \cos(\theta_{ef} - \frac{2\pi}{3})$$

$$f_{cs} = \sqrt{2} f_s \cos(\theta_{ef} + \frac{2\pi}{3})$$

$$f_{as} + f_{bs} + f_{cs} = 0 \text{ (balanced set)}$$

$$\theta_{ef} = \int_0^t \omega_e(t)dt + \theta_{ef}(0)$$

θ_{ef} Angular position of each electrical variable (voltage, current, and flux linkage) is θ_{ef} with the f subscript used to denote the specific electrical variable.

θ_e: Angular position of the synchronously rotating reference frame is θ_e. θ_e and θ_e differ only in the zero position $\theta_e(0)$ and $\theta_{ef}(0)$, since each has the same angular velocity of ω_e.f_{as}, f_{bs} and f_{cs} can be transformed to the arbitrary reference frame,

$$\bar{f}_{qd0s} = \overline{\mathbf{K}}_s \bar{f}_{abcs}$$

After transformation, we will have,

$$f_{qs} = \sqrt{2} f_s \cos(\theta_{ef} - \theta)$$

$$f_{ds} = -\sqrt{2} f_s \sin(\theta_{ef} - \theta)$$

$$f_{0s} = 0$$

qs and ds variables form a balanced 2-phase set in all reference frames except when $\omega=\omega_e$,

$$f^e{}_{qs} = \sqrt{2} f_s \cos\left[\theta_{ef}(0) - \theta_e(0)\right]$$
$$f^e{}_{ds} = -\sqrt{2} f_s \sin\left[\theta_{ef}(0) - \theta_e(0)\right]$$

In qs^e and ds^e reference frame, sinusoidal quantities appear as constant dc quantities.

Balanced Steady-State Phasor Relationships

For balanced steady-state conditions ωe is constant and sinusoidal quantities can be represented as phasor variables.

$$F_{as} = \sqrt{2}F_s \cos\left[\omega_e t + \theta_{ef}(0)\right] = \mathrm{Re}\left[\sqrt{2}F_s e^{j\theta_{ef}(0)} e^{j\omega_e t}\right]$$
$$F_{bs} = \sqrt{2}F_s \cos\left[\omega_e t + \theta_{ef}(0) - \frac{2\pi}{3}\right] = \mathrm{Re}\left[\sqrt{2}F_s e^{j\left(\theta_{ef}(0) - \frac{2\pi}{3}\right)} e^{j\omega_e t}\right]$$
$$F_{bs} = \sqrt{2}F_s \cos\left[\omega_e t + \theta_{ef}(0) + \frac{2\pi}{3}\right] = \mathrm{Re}\left[\sqrt{2}F_s e^{j\left(\theta_{ef}(0) + \frac{2\pi}{3}\right)} e^{j\omega_e t}\right]$$

Balanced steady-state qs-ds variables are,

$$F_{qs} = \sqrt{2}F_s \cos\left[(\omega_e - \omega)t + \theta_{ef}(0) - \theta(0)\right]$$
$$= \mathrm{Re}\left[\sqrt{2}F_s e^{j(\theta_{ef}(0) - \theta(0))} e^{j(\omega_e - \omega)t}\right]$$

$$F_{ds} = -\sqrt{2}F_s \sin\left[(\omega_e - \omega)t + \theta_{ef}(0) - \theta(0)\right]$$
$$= \mathrm{Re}\left[j\sqrt{2}F_s e^{j(\theta_{ef}(0) - \theta(0))} e^{j(\omega_e - \omega)t}\right]$$

f_{as} phasor can be expressed as

$$\tilde{F}_{as} = F_s e^{j\theta_{ef}(0)}$$

For arbitrary reference frame ($\omega \neq \omega e$),

$$\tilde{F}_{qs} = F_s e^{j(\theta_{ef}(0) - \theta(0))}, \quad \tilde{F}_{ds} = j\tilde{F}_{qs}$$

Selecting $\theta(0)=0$,

$$\tilde{F}_{as} = \tilde{F}_{qs}$$

Thus, in all asynchronously rotating reference frame ($\omega \neq \omega_e$) with $\theta(0) = 0$, the phasor representing the as variables is equal to the phasor representing the qs variables. In the synchronously rotating reference frame $\omega=\omega_e$, $F^e{}_{qs}$ and $F^e{}_{ds}$ can be expressed as

$$F^e{}_{qs} = \mathrm{Re}\left[\sqrt{2}F_s e^{j(\theta_{ef}(0) - \theta(0))}\right]$$

$$F^e{}_{ds} = \mathrm{Re}\left[j\sqrt{2}F_s e^{j(\theta_{ef}(0) - \theta(0))}\right]$$

Let $\theta_e(0)=0$, then

$$F^e{}_{qs} = \sqrt{2}F_s \cos(\theta_{ef}(0)),\ F^e{}_{ds} = -\sqrt{2}F_s \sin(\theta_{ef}(0))$$

$$\sqrt{2}\widetilde{F}_{as} = F^e{}_{qs} - jF^e{}_{ds}$$

since, $\widetilde{F}_{as} = F_s e^{j(\theta_{ef}(0))} = F_s \cos(\theta_{ef}(0)) + jF_s \sin(\theta_{ef}(0))$