3-Phase Induction Machines - Dynamic Modeling Using Reference Frame Theory

Winding arrangement for a 2-pole, 3-phase, wye-connected symmetrical induction machine is shown in Fig.1. Stator windings are identical, sinusoidally distributed windings, displaced by120°, with Ns equivalent turns and resistance r_s. Consider the case when rotor windings are also three identical sinusoidally distributed windings, displaced by120°, with Nr equivalent turns and resistance r_r.

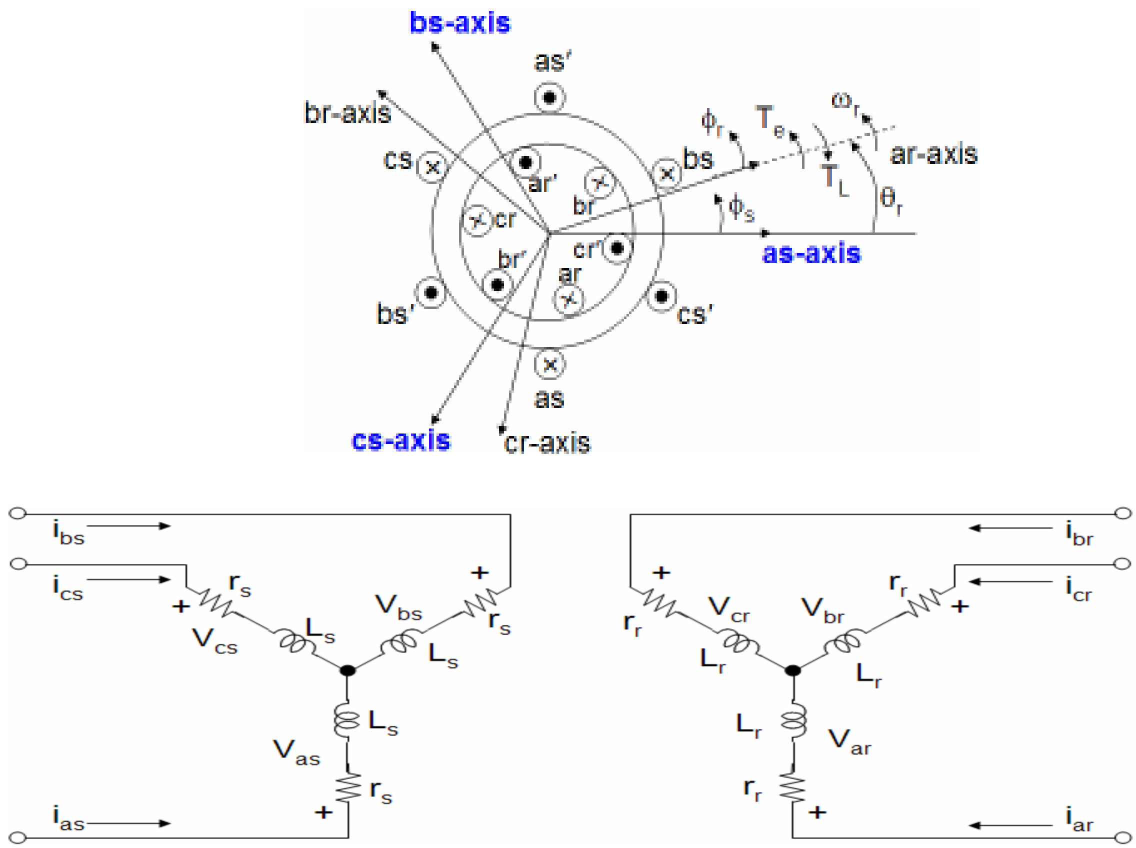


Fig1. 2 pole Three Phase Induction Machine

In abc reference frame, voltage equations can be written as

$$V_{abcs} = r_s i_{abcs} + p\lambda_{abcs}$$
$$V_{abcr} = r_r i_{abcr} + p\lambda_{abcr}$$
$$(f_{abcs})^T = [f_{as} \quad f_{bs} \quad f_{cs}], \quad (f_{abcr})^T = [f_{ar} \quad f_{br} \quad f_{cr}]$$

s: denotes variables and parameters associated with the stator circuits and r: denotes variables and parameters associated with the rotor circuits.

$$\begin{bmatrix} \lambda_{abcs} \\ \lambda_{abcr} \end{bmatrix} = \begin{bmatrix} L_s & L_{sr} \\ (L_{sr})^T & L_r \end{bmatrix} \begin{bmatrix} i_{abcs} \\ i_{abcr} \end{bmatrix}$$

Where,

$$\mathbf{L}_s = \begin{bmatrix} L_{ls} + L_{ms} & -\frac{1}{2}L_{ms} & -\frac{1}{2}L_{ms} \\ -\frac{1}{2}L_{ms} & L_{ls} + L_{ms} & -\frac{1}{2}L_{ms} \\ -\frac{1}{2}L_{ms} & -\frac{1}{2}L_{ms} & L_{ls} + L_{ms} \end{bmatrix}, \qquad \mathbf{L}_r = \begin{bmatrix} L_{lr} + L_{mr} & -\frac{1}{2}L_{mr} & -\frac{1}{2}L_{mr} \\ -\frac{1}{2}L_{mr} & L_{lr} + L_{mr} & -\frac{1}{2}L_{mr} \\ -\frac{1}{2}L_{mr} & -\frac{1}{2}L_{mr} & L_{lr} + L_{mr} \end{bmatrix}$$

L_{ls} and L_{ms} are, respectively, the leakage and magnetizing inductance of the stator windings. L_{lr} and L_{mr} are, respectively, the leakage and magnetizing inductance of the rotor windings.

$$\mathbf{L}_{sr} = \mathbf{L}_{rs} = L_{ms} \begin{bmatrix} \cos\theta_r & \cos(\theta_r + \frac{2\pi}{3}) & \cos(\theta_r - \frac{2\pi}{3}) \\ \cos(\theta_r - \frac{2\pi}{3}) & \cos\theta_r & \cos(\theta_r + \frac{2\pi}{3}) \\ \cos(\theta_r + \frac{2\pi}{3}) & \cos(\theta_r - \frac{2\pi}{3}) & \cos\theta_r \end{bmatrix},$$

"L_{sr}" is the amplitude of the mutual inductances between stator and rotor windings. A majority of induction machines are not equipped with coil-wound rotor windings; instead, the current flows in copper or aluminum bars which are uniformly distributed in a common ring at each end of the rotor. This type of rotor is referred to as a squirrel-cage rotor. Rotor variables can be referred to the stator windings by appropriate turn's ratio.

$$i'_{abcr} = \frac{N_r}{N_s} i_{abcr}, \quad V'_{abcr} = \frac{N_s}{N_r} V_{abcr}, \quad \lambda'_{abcr} = \frac{N_s}{N_r} \lambda_{abcr}, \quad L_{ms} = \left(\frac{N_s}{N_r}\right)^2 L_{sr}$$

$$[\mathbf{L}'_{sr}] = \frac{N_s}{N_r}[\mathbf{L}_{sr}] = L_{ms} \begin{bmatrix} \cos\theta_r & \cos(\theta_r + \frac{2\pi}{3}) & \cos(\theta_r - \frac{2\pi}{3}) \\ \cos(\theta_r - \frac{2\pi}{3}) & \cos\theta_r & \cos(\theta_r + \frac{2\pi}{3}) \\ \cos(\theta_r + \frac{2\pi}{3}) & \cos(\theta_r - \frac{2\pi}{3}) & \cos\theta_r \end{bmatrix},$$

Also,

$$L_{mr} = \left(\frac{N_r}{N_s}\right)^2 L_{ms}, \quad [\mathbf{L}'_r] = \left(\frac{N_r}{N_s}\right)^2 [\mathbf{L}_r]$$

$$[\mathbf{L}'_r] = \begin{bmatrix} L'_{lr} + L_{mr} & -\frac{1}{2}L_{ms} & -\frac{1}{2}L_{ms} \\ -\frac{1}{2}L_{ms} & L_{lr} + L_{ms} & -\frac{1}{2}L_{ms} \\ -\frac{1}{2}L_{ms} & -\frac{1}{2}L_{ms} & L_{lr} + L_{ms} \end{bmatrix}$$

Where,

$$L'_{lr} = \left(\frac{N_s}{N_r}\right)^2 L_{lr}$$

Flux linkage may be expressed as

$$\begin{bmatrix} \lambda_{abcs} \\ \lambda'_{abcr} \end{bmatrix} = \begin{bmatrix} \mathbf{L}_s & \mathbf{L}'_{sr} \\ (\mathbf{L}'_{sr})^T & \mathbf{L}'_r \end{bmatrix} \begin{bmatrix} i_{abcs} \\ i'_{abcr} \end{bmatrix}$$

Voltage equations expressed in terms of machine variables referred to the stator windings may be written as

$$\begin{bmatrix} V_{abcs} \\ V'_{abcr} \end{bmatrix} = \begin{bmatrix} \mathbf{r}_s + p\mathbf{L}_s & p\mathbf{L}'_{sr} \\ p(\mathbf{L}'_{sr})^T & \mathbf{r}'_r + p\mathbf{L}'_r \end{bmatrix} \begin{bmatrix} i_{abcs} \\ i'_{abcr} \end{bmatrix}$$

Where,

$$r'_r = \left(\frac{N_s}{N_r}\right)^2 r_r$$

Energy stored in the coupling field may be written as

$$W_c = W_f = \frac{1}{2}(i_{abcs})^T(\mathbf{L}_s - \mathbf{L}_{ls}\mathbf{I})i_{abcs} +$$
$$(i_{abcs})^T(\mathbf{L}_{sr})i'_{abcr} + \frac{1}{2}(i'_{abcr})^T(\mathbf{L}'_r - \mathbf{L}'_{lr}\mathbf{I})i'_{abcr}$$

Where, **I**: identity matrix

Voltage equations expressed in terms of machine variables referred to the stator windings may be written as

$$T_e(i_j, \theta_r) = \frac{P}{2}\frac{\partial W_c(i_j, \theta_r)}{\partial \theta_r}$$

Since $\mathbf{L}_s$ and $\mathbf{L}_r$ are functions of θ_r, the above equation for the electromagnetic torque yields.

$$T_e = (\frac{p}{2})(i_{abcs})^T \frac{\partial}{\partial \theta_r}[\mathbf{L}'_{sr}]i'_{abcr}$$

$$= -\frac{P}{2}L_{ms}\left\{\begin{aligned}&\left[i_{as}(i'_{ar} - \frac{1}{2}i'_{br} - \frac{1}{2}i'_{cr}) + i_{bs}(i'_{br} - \frac{1}{2}i'_{ar} - \frac{1}{2}i'_{cr}) + i_{cs}(i'_{cr} - \frac{1}{2}i'_{br} - \frac{1}{2}i'_{ar})\right]\sin\theta_r \\ &+ \frac{\sqrt{3}}{2}\left[i_{as}(i'_{br} - i'_{cr}) + i_{bs}(i'_{cr} - i'_{ar}) + i_{cs}(i'_{ar} - i'_{br})\right]\cos\theta_r\end{aligned}\right\}$$

The torque and rotor speed are related by

$$T_e = J\left(\frac{2}{P}\right)p\omega_r + T_L$$

Equations of Transformation for Rotor Circuit

In the analysis of induction machines it is desirable to transform the variables associated with the symmetrical rotor windings to the arbitrary reference frame.

$$f'_{qd0r} = \mathbf{K}_r f'_{abcr}$$
$$(f'_{qd0r})^T = \left[f'_{qr} \quad f'_{dr} \quad f'_{0r}\right]$$
$$(f'_{abcr})^T = \left[f'_{ar} \quad f'_{br} \quad f'_{cr}\right]$$

$$\mathbf{K}_r = \frac{2}{3}\begin{bmatrix}\cos\beta & \cos(\beta - \frac{2\pi}{3}) & \cos(\beta + \frac{2\pi}{3}) \\ \sin\beta & \sin(\beta - \frac{2\pi}{3}) & \sin(\beta + \frac{2\pi}{3}) \\ \frac{1}{2} & \frac{1}{2} & \frac{1}{2}\end{bmatrix},$$

where, β=θ-θr from figure below

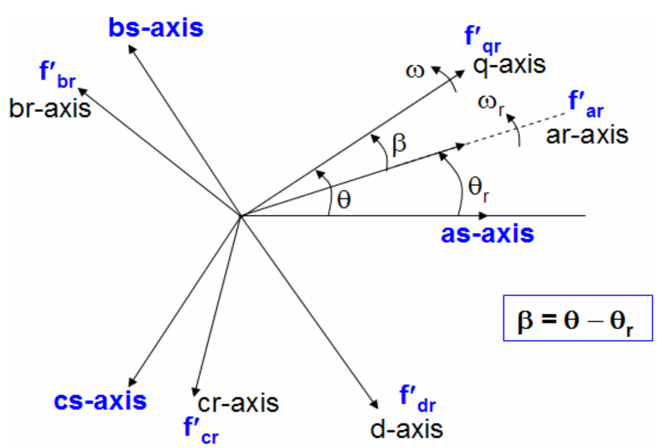


Fig. 2. Axis of 2-pole, 3-phase Symmetrical machine.

$$\theta_r = \int_0^t \omega_r(t)dt + \theta_r(0)$$

$$(\mathbf{K}_r)^{-1} = \begin{bmatrix} \cos\beta & \sin\beta & 1 \\ \cos(\beta - \frac{2\pi}{3}) & \sin(\beta - \frac{2\pi}{3}) & 1 \\ \cos(\beta + \frac{2\pi}{3}) & \sin(\beta + \frac{2\pi}{3}) & 1 \end{bmatrix}$$

"r" subscript indicates the variable, parameters and transformation associated with rotating circuits.

Voltage Equations in Arbitrary Reference Frame Variables

For two-pole, 3-phase symmetrical induction,

$$\overline{V}_{abcs} = \overline{r}_s i_{abcs} + p\lambda_{abcs} \qquad \lambda_{abcs} = (\overline{L}_s) i_{abcs} + (L'_{sr}) i'_{abcr} \qquad \overline{V}_{abcs} = \mathbf{K}_s \overline{V}_{qd0s}, \quad i_{abcs} = \mathbf{K}_s \bar{i}_{qd0s}$$

$$V'_{abcr} = r'_r i'_{abcr} + p\lambda'_{abcr} \qquad \lambda'_{abcr} = (\overline{L}'_{sr})^T i_{abcs} + (L'_r) i'_{abcr} \qquad V'_{abcr} = \mathbf{K}_r \overline{V}'_{qd0r}, \quad i'_{abcr} = \mathbf{K}_r \bar{i}'_{qd0s}$$

Using the above transformation equations, we can transform the voltage equations to an arbitrary reference frame rotating at speed of ω.

$$V_{qd0s} = r_s i_{qd0s} + \omega\lambda_{qds} + p\lambda_{qd0s}$$

$$V'_{qd0r} = r'_r i_{qd0r} + (\omega - \omega_r)\lambda'_{qdr} + p\lambda'_{qd0r}$$

$$where, \quad (\lambda_{qds})^T = [\lambda_{ds} \quad -\lambda_{qs} \quad 0], \quad (\lambda'_{qdr})^T = [\lambda'_{dr} \quad -\lambda'_{qr} \quad 0]$$

Flux linkage equations in abc reference frame can be transformed to qd axes using K_s and K_r transformation matrices.

$$\begin{bmatrix} \lambda_{qd0s} \\ \lambda'_{qd0r} \end{bmatrix} = \begin{bmatrix} \mathbf{K}_s \mathbf{L}_s (\mathbf{K}_s)^{-1} & \mathbf{K}_s \mathbf{L}'_{sr} (\mathbf{K}_r)^{-1} \\ \mathbf{K}_r \mathbf{L}'_{sr} (\mathbf{K}_s)^{-1} & \mathbf{K}_r \mathbf{L}'_r (\mathbf{K}_r)^{-1} \end{bmatrix} \begin{bmatrix} i_{qd0s} \\ i'_{qd0r} \end{bmatrix}$$

Where

$$\mathbf{K}_s \mathbf{L}_s (\mathbf{K}_s)^{-1} = \begin{bmatrix} L_{ls} + M & 0 & 0 \\ 0 & L_{ls} + M & 0 \\ 0 & 0 & L_{ls} + M \end{bmatrix}, \quad M = \frac{3}{2} L_{ms}$$

$$\mathbf{K}_r \mathbf{L}'_r (\mathbf{K}_r)^{-1} = \begin{bmatrix} L'_{lr} + M & 0 & 0 \\ 0 & L'_{lr} + M & 0 \\ 0 & 0 & L'_{lr} + M \end{bmatrix}, \quad M = \frac{3}{2} L_{ms}$$

$$\mathbf{K}_s\mathbf{L}'_{sr}\left(\mathbf{K}_r\right)^{-1} = \mathbf{K}_r\left(\mathbf{L}'_{sr}\right)^T\left(\mathbf{K}_s\right)^{-1} = \begin{bmatrix} M & 0 & 0 \\ 0 & M & 0 \\ 0 & 0 & M \end{bmatrix}$$

Voltage equations written in expanded form can be expressed as

$$V_{qs} = r_s i_{qs} + \omega\lambda_{ds} + p\lambda_{qs} \qquad V'_{qr} = r'_r i'_{qr} + (\omega - \omega_r)\lambda'_{dr} + p\lambda'_{qr}$$
$$V_{ds} = r_s i_{ds} - \omega\lambda_{qs} + p\lambda_{ds}, \qquad V'_{dr} = r'_r i'_{dr} - (\omega - \omega_r)\lambda'_{qr} + p\lambda'_{dr}$$
$$V_{0s} = r_s i_{0s} + p\lambda_{0s} \qquad V'_{0r} = r'_r i'_{0r} + p\lambda'_{0r}$$

Flux linkage equations are

$$\lambda_{qs} = L_{ls} i_{qs} + M\left(i_{qs} + i'_{qr}\right) \qquad \lambda'_{qr} = L'_{lr} i'_{qr} + M\left(i_{qs} + i'_{qr}\right)$$
$$\lambda_{ds} = L_{ls} i_{ds} + M\left(i_{ds} + i'_{dr}\right) \qquad \lambda'_{dr} = L'_{lr} i'_{dr} + M\left(i_{ds} + i'_{dr}\right)$$
$$\lambda_{0s} = L_{ls} i_{0s} \qquad \lambda'_{0r} = L'_{lr} i'_{0r}$$

Since machine and power system parameters are nearly always given in ohms or percent or per unit of a base impedance, it is convenient to express the voltage and flux linkage equations in terms of reactances rather than inductances.

Let

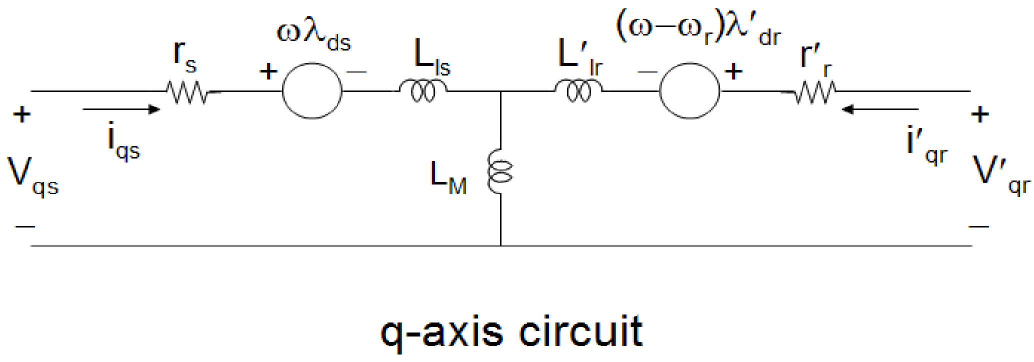
$$\varphi = \lambda\omega_b$$

Then

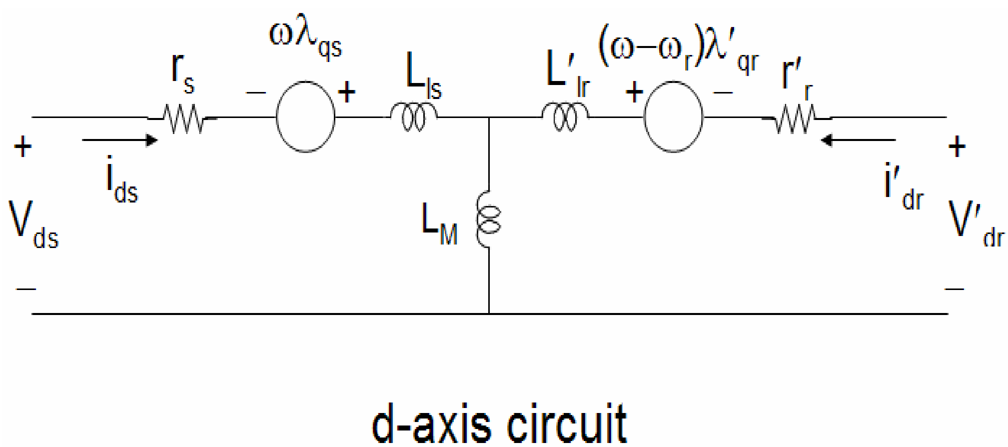
$$V_{qs} = r_s i_{qs} + \frac{\omega}{\omega_b}\varphi_{ds} + p\varphi_{qs} \qquad V'_{qr} = r'_r i'_{qr} + \frac{(\omega - \omega_r)}{\omega_b}\varphi'_{dr} + \frac{p}{\omega_b}\varphi'_{qr}$$
$$V_{ds} = r_s i_{ds} - \frac{\omega}{\omega_b}\varphi_{qs} + \frac{p}{\omega_b}\varphi_{ds}, \qquad V'_{dr} = r'_r i'_{dr} - \frac{(\omega - \omega_r)}{\omega_b}\varphi'_{qr} + \frac{p}{\omega_b}\varphi'_{dr}$$
$$V_{0s} = r_s i_{0s} + \frac{p}{\omega_b}\varphi_{0s} \qquad V'_{0r} = r'_r i'_{0r} + \frac{p}{\omega_b}\varphi'_{0r}$$

And flux linkages become flux linkages per second with the units of volts.

$$\varphi_{qs} = X_{ls} i_{qs} + X_m\left(i_{qs} + i'_{qr}\right) \qquad \varphi'_{qr} = X'_{lr} i'_{qr} + X_m\left(i_{qs} + i'_{qr}\right)$$
$$\varphi_{ds} = X_{ls} i_{ds} + X_m\left(i_{ds} + i'_{dr}\right), \qquad \varphi'_{dr} = X'_{lr} i'_{dr} + X_m\left(i_{ds} + i'_{dr}\right)$$
$$\varphi_{0s} = X_{ls} i_{0s} \qquad \varphi'_{0r} = X'_{lr} i'_{0r}$$



Equivalent circuits of a 3-phase, symmetrical induction machine with rotating q-d axis at speed of ω.



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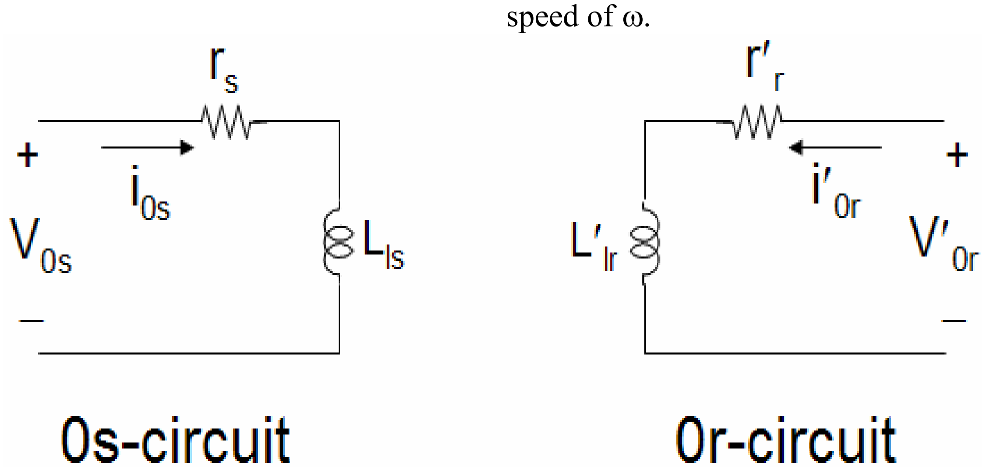

Equivalent circuits of a 3-phase, symmetrical induction machine with rotating q-d axis at speed of ω.

Electromagnetic torque in terms of arbitrary reference frame variables may be obtained by substituting the equations of transformation in

$$T_e = \frac{P}{2}\left(i_{abcs}\right)^T \frac{\partial}{\partial \theta_r}\left(L'_{sr}\right) i'_{abcr}$$

$$= \frac{P}{2}\left[(\mathbf{K}_s)^{-1} i_{qd0s}\right]^T \frac{\partial}{\partial \theta_r}\left(L'_{sr}\right)(\mathbf{K}_r)^{-1} i'_{qd0r}$$

After some work, we will have the following:

$$T_e = \left(\frac{3}{2}\right)\left(\frac{P}{2}\right) M\,(i_{qs} i'_{dr} - i_{ds} i'_{qr})$$

Where, Te is positive for motor action. Other expressions for the electromagnetic torque of an induction machine are

$$T_e = \left(\frac{3}{2}\right)\left(\frac{P}{2}\right)(\lambda'_{qr} i'_{dr} - \lambda'_{dr} i'_{qr})$$

$$T_{em} = \left(\frac{3}{2}\right)\left(\frac{P}{2}\right)(\lambda_{ds} i_{qs} - \lambda_{qs} i_{ds})$$

$$T_e = \left(\frac{3}{2}\right)\left(\frac{P}{2}\right)\left(\frac{1}{\omega_b}\right)(\varphi'_{qr} i'_{dr} - \varphi'_{dr} i'_{qr})$$