UNIT-2 (Additional Problems)

Test whether the following polynomials are Hurwitz

1. $p(s) = s^3 + 4s^2 + 5s + 2$

$m(s) = 4s^2 + 2$ & $n(s) = s^3 + 5s$

$$\psi(s) = \frac{n(s)}{m(s)} = \frac{s^3 + 5s}{4s^2 + 2}$$

$4s^2+2 \,)\, s^3 + 5s \,(\, \frac{s}{4}$

$\quad s^3 + \frac{s}{2}$

$\quad \overline{\frac{9}{2}s \,)\, 4s^2 + 2 \,(\, \frac{8}{9}s}$

$\qquad 4s^2$

$\qquad \overline{2 \,)\, \frac{9}{2}s \,(\, \frac{9}{4}s}$

$\qquad\quad \frac{9}{2}s$

$\qquad\quad \overline{0}$

$$\psi(s) = \frac{s}{4} + \cfrac{1}{\frac{8}{9}s + \cfrac{1}{\frac{9}{4}s}}$$

Since all coefficients are positive and real $p(s)$ is Hurwitz.

2. $p(s) = s^4 + 7s^3 + 6s^2 + 21s + 8$

$m(s) = s^4 + 6s^2 + 8$ & $n(s) = 7s^3 + 21s$

$\psi(s) = \frac{m(s)}{n(s)} \Rightarrow 7s^3 + 21s \,)\, s^4 + 6s^2 + 8 \,(\, \frac{s}{7}$

$\quad s^4 + 3s^2$

$\quad \overline{3s^2 + 8 \,)\, 7s^3 + 21s \,(\, \frac{7}{3}s}$

$\qquad 7s^3 + \frac{56}{3}s$

$\qquad \overline{\frac{7}{3}s \,)\, 3s^2 + 8 \,(\, \frac{9}{7}s}$

$\qquad\quad 3s^2$

$\qquad\quad \overline{8 \,)\, \frac{7}{3}s \,(\, \frac{7}{24}s}$

$\qquad\qquad \frac{7}{3}s$

$\qquad\qquad \overline{0}$

$$\psi(s) = \frac{s}{7} + \cfrac{1}{\frac{7}{3}s + \cfrac{1}{\frac{9}{7}s + \cfrac{1}{\frac{7}{24}s}}}$$

All the coefficients are positive $p(s)$ is Hurwitz.

3. $p(s) = s^7 + 2s^6 + 2s^5 + s^4 + 4s^3 + 8s^2 + 8s + 4$

$m(s) = 2s^6 + s^4 + 8s^2 + 4$ & $n(s) = s^7 + 2s^5 + 4s^3 + 8s$

$\psi = \frac{n(s)}{m(s)}$

$$2s^6+s^4+8s^2+4 \;)\; s^7 + 2s^5 + 4s^3 + 8s \;(\tfrac{s}{2}$$
$$s^7 + \frac{s^5}{2} + 4s^3 + 2s$$
$$\frac{3}{2}s^5 + 6s \;)\; 2s^6 + s^4 + 8s^2 + 4 \;(\tfrac{4}{3}s$$
$$2s^6 + 0 + 8s^2$$
$$s^4 + 4 \;)\; \frac{3}{2}s^5 + 6s \;(\tfrac{3}{2}s$$
$$\frac{3}{2}s^5 + 6s$$
$$0$$

The expansion terminates abruptly

Hence we examine $p_1(s) = s^4 + 4$ as before it is not Hurwitz. Hence $p(s)$ is not Hurwitz.

4. $p(s) = s^4 + s^3 + 2s^2 + 3s + 2$

$m(s) = s^4 + 2s^2 + 2$ & $n(s) = s^3 + 3s$

$\psi = \frac{n(s)}{m(s)}$

$$s^4+2s^2+2 \;)\; s^3 + 2s \;(\tfrac{1}{s}$$
$$s^3 + 2s$$
$$0$$

Not Hurwitz

5. $F(s) = s^3 + 2s^2 + 3s + 6$

s^3	1	3
s^2	2	6
s^1	$\frac{6-6}{2} = 0$	0
s^0	0	0

Since the row vanishes $F_1(s)$ is formed by the row just above the vanishing row.

$F_1(s) = 2s^2 + 6 \quad \therefore F_1'(s) = 4s$

s^3	1	3
s^2	2	6
s^1	4	0
s^0	$\frac{24-0}{4} = 6$	0

All terms are real & positive. F(s) is Hurwitz.

6. $F(s) = s^5 + 3s^3 + s$

There are missing terms in F(s)

find $F'(s) = 5s^4 + 9s^2 + 1$

s^5	1	3	1
s^4	5	9	1
s^3	$\frac{15-9}{5} = \frac{6}{5}$	$\frac{5-1}{5} = \frac{4}{5}$	0
s^2	$\frac{17}{3}$	1	0
s^1	$\frac{10}{17}$	0	0
s^0	1	0	0

All coefficients in first column are real and positive. F(s) is Hurwitz.

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$5s^4+9s^2+1 \,\big)\, s^5+3s^3+s \,\big(\, \frac{s}{5}$

$s^5+\frac{9s^3}{5}+\frac{s}{5}$

$\frac{6s^3}{5}+\frac{4s}{5} \,\big)\, 5s^4+9s^2+1 \,\big(\, \frac{25s}{6}$

$5s^4+\frac{10s^2}{3}$

$\frac{17s^2}{3}+1 \,\big)\, \frac{6s^3}{5}+\frac{4s}{5} \,\big(\, \frac{18}{85}s$

$\frac{6s^3}{5}+\frac{18}{85}s$

$\frac{10}{17}s \,\big)\, \frac{17s^2}{3}+1 \,\big(\, \frac{289}{30}s$

$\frac{17s^2}{3}$

1

F(s) is Hurwitz.

7. $F(s) = s^4+5s^3+11s^2+25s+6$

solution:

$m(s) = s^4+11s^2+6 \qquad n(s) = 5s^3+25s$

$5s^3+25s \,\big)\, s^4+11s^2+6 \,\big(\, \frac{s}{5}$

s^4+5s^2

$6s^2+6 \,\big)\, 5s^3+25s \,\big(\, \frac{5s}{6}$

$5s^3+5s$

$20s \,\big)\, 6s^2+6 \,\big(\, \frac{6}{20}s$

$6s^2$

6

All coefficients are real and positive

Hence Hurwitz.

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8. $F(s) = s^6 + 7s^4 + 14s^2 + 8$

Solution:

All odd degree terms are missing.

$\therefore F'(s) = 6s^5 + 28s^3 + 28s$

$$6s^5+28s^3+28s \,\big)\, s^6 + 7s^4 + 14s^2 + 8 \,\big(\, \frac{s}{6}$$

$$\underline{s^6 + 4.67s^4 + 4.67s^2}$$

$$2.33s^4 + 9.33s^2 + 8 \,\big)\, 6s^5 + 28s^3 + 28s \,\big(\, \frac{6s}{2.33}$$

$$\underline{6s^5 + 24s^3 + 20.6s}$$

$$4s^3 + 7.4s \,\big)\, 2.33s^4 + 9.33s^2 + 8 \,\big(\, \frac{2.33s}{4}$$

$$\underline{2.33s^4 + 4.31s^2}$$

$$5.02s^2 + 8 \,\big)\, 4s^3 + 7.4s \,\big(\, \frac{4s}{5.02}$$

$$\underline{4s^3 + 6.37s}$$

$$1.03s \,\big)\, 5.02s^2 + 8 \,\big(\, \frac{5.02s}{1.03}$$

$$\underline{5.02s^2 +}$$

$$\underline{8}$$

All coefficients are real and positive

Hence Hurwitz.

9. $p(s) = s^3 + 4s^2 + 5s + 2$

Solution:

$m(s) = 4s^2 + 2 \quad n(s) = s^3 + 5s$

$$4s^2+2)\, s^3 + 5s \,\left(\frac{s}{4}\right.$$

$$s^3 + 0.5s$$

$$4.5s\,)\, 4s^2 + 2 \,\left(\frac{4s}{4.5}\right.$$

$$4s^2 + \cancel{2}$$

$$2$$

All coefficients are real and positive

Hence Hurwitz.

10. $F(s) = s^5 + 2s^4 + 5s^3 + 10s^2 + 9s + 18$

Solution:

$m(s) = 2s^4 + 10s^2 + 18 \quad n(s) = s^5 + 5s^3 + 9s$

$$2s^4 + 10s^2 + 18)\, s^5 + 5s^3 + 9s \,\left(\frac{s}{2}\right.$$

$$s^5 + 5s^3 + 9s$$

$$0$$

Terminates abruptly.

$\therefore F(s) = 2s^4 + 10s^2 + 18 \quad F'(s) = 8s^3 + 20s$

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Positive Real function

11. $F(s) = \dfrac{2s^4+7s^3+11s^2+12s+4}{s^4+5s^3+9s^2+11s+6}$

Solution:

1. $p(s)$ & $q(s)$ must be Hurwitz

$p(s) = 2s^4+7s^3+11s^2+12s+4$, $q(s) = s^4+5s^3+9s^2+11s+6$

$m(s) = 2s^4+11s^2+4$ $\qquad$ $m(s) = s^4+9s^2+6$

$n(s) = 7s^3+12s$ $\qquad$ $n(s) = 5s^3+11s$

$7s^3+12s\ )\ 2s^4+11s^2+4\ \left(\frac{2s}{7}\right.$

$\quad 2s^4+\frac{24}{7}s^2$

$\quad \frac{53}{7}s^2+4\ )\ 7s^3+12s\ \left(\frac{49}{53}s\right.$

$\quad\quad 7s^3+3.7s$

$8.3s\ )\ \frac{53}{7}s^2+4\ (0.945s$

$\quad \frac{53}{7}s^2$

$\quad 4\ )\ 8.3s\ \left(\frac{8.3s}{4}\right.$

$\quad\quad 8.3s$

$\quad\quad 0$

$5s^3+11s\ )\ s^4+9s^2+6\ \left(\frac{s}{5}\right.$

$\quad s^4+\frac{11}{5}s^2$

$\quad \frac{34}{5}s^2+6\ )\ 5s^3+11s\ \left(\frac{25}{34}s\right.$

$\quad\quad 5s^3+4.4s$

$6.6s\ )\ \frac{34}{5}s^2+6\ \left(\frac{6.8}{6.6}s\right.$

$\quad \frac{34}{5}s^2$

$\quad 6\ )\ 6.6s\ \left(\frac{6.6}{6}s\right.$

$\quad\quad 6.6s$

$\quad\quad 0$

Hurwitz

a

$p(s)$ & $q(s)$ is Hurwitz

2. Residues of imaginary axis poles must be real and positive

$q(s) = s^4+5s^3+9s^2+11s+6$

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3. $A(\omega^2) \geq 0$ for $0 < \omega < \infty$

$$A(\omega^2) = m_1(s)\, m_2(s) - n_1(s)\, n_2(s) \Big|_{s=j\omega}$$

$$p(s) = 2s^4 + 7s^3 + 11s^2 + 12s + 4$$

$$\hookrightarrow m_1(s) = 2s^4 + 11s^2 + 4 \qquad n_1(s) = 7s^3 + 12s$$

$$q(s) = s^4 + 5s^3 + 9s^2 + 11s + 6$$

$$\hookrightarrow m_2(s) = s^4 + 9s^2 + 6 \qquad n_2(s) = 5s^3 + 11s$$

$$A(\omega^2) = \left[(2s^4 + 11s^2 + 4)(s^4 + 9s^2 + 6)\right] - \left[(7s^3 + 12s)\cdot(5s^3 + 11s)\right] \Big|_{s=j\omega}$$

$$= (2s^8 + \cancel{11} s^6 + 4s^4 + 18s^6 + 99s^4 + 36s^2 + 12s^4 + 66s^2 +$$
$$24 - 35s^6 - 77s^4 - 60s^4 - 132s^2) \Big|_{s=j\omega}$$

$$= (2s^8 - 6s^6 - 22s^4 - 30s^2 + 24) \Big|_{s=j\omega}$$

$$= 2(j\omega)^8 - 6(j\omega)^6 - 22(j\omega)^4 - 30(j\omega)^2 + 24$$

$$A(\omega^2) = 2\omega^8 + 6\omega^6 - 22\omega^4 + 30\omega^2 + 24$$

$$A(\omega^2) = 2 + 6 - 22 + 30 + 24 > 0 \qquad \text{At } \omega = 1$$

$A(\omega^2) > 0$ for all ω.

Since all conditions are satisfied the function is PRF.

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12. $F(s) = \frac{s^2+6s+5}{s^2+9s+14}$

Solution:

1. p(s) & q(s) must be Hurwitz

$p(s) = s^2+6s+5$

$q(s) = s^2+9s+14$

$$6s\,)\,s^2+5\,(\tfrac{s}{6}$$
$$\underline{s^2}$$
$$5\,)\,6s\,(\tfrac{6}{5}s$$
$$\underline{6s}$$
$$0$$

$$9s\,)\,s^2+14\,(\tfrac{s}{9}$$
$$\underline{s^2}$$
$$14\,)\,9s\,(\tfrac{9s}{14}$$
$$\underline{9s}$$
$$0$$

p(s) ⇒ Hurwitz

q(s) ⇒ Hurwitz

2. Residues at imaginary axis poles must be positive & real.

$q(s) = s^2+9s+14$

$= (s+7)(s+2)$

There are no poles on imaginary axis

Condition satisfied.

3. $A(\omega^2) \geq 0$ for $0 \leq \omega < \infty$.

$A(\omega^2) = m_1(s)m_2(s) - n_1(s)\,n_2(s)\big|_{s=j\omega}$

$p(s) = s^2+6s+5$, $m_1(s) = s^2+5$ $n_1(s) = 6s$

$q(s) = s^2+9s+14$, $m_2(s) = s^2+14$, $n_2(s) = 9s$

$A(\omega^2) = (s^2+5)(s^2+14) - (6s \times 9s)\big|_{s=j\omega}$

$= s^4+5s^2+14s^2+70 - 54s^2\big|_{s=j\omega}$

$= s^4 - 35s^2+70\big|_{s=j\omega}$

$A(\omega^2) = (j\omega)^4 - 35(j\omega)^2+70$

$= \omega^4+35\omega^2+70$

At $\omega=1$, $A(\omega^2) = 1+35+70 \geq 0$

The function is PRF

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13. $F(s) = \dfrac{s^2 + \frac{3}{4}s + \frac{3}{4}}{s^2 + s + 4}$

Solution:

1. p(s) & q(s) must be Hurwitz

$p(s) = s^2 + \frac{3}{4}s + \frac{3}{4}$ $\qquad\qquad$ $q(s) = s^2 + s + 4$

$\frac{3}{4}s \,)\, s^2 + \frac{3}{4} \,(\, \frac{4s}{3}$

$\quad s^2$

$\frac{3}{4} \,)\, \frac{3}{4}s \,(\, s$

$\quad \frac{3}{4}s$

$\quad 0$

$s \,)\, s^2 + 4 \,(\, s$

$\quad s^2$

$4 \,)\, s \,(\, \frac{s}{4}$

$\quad s$

$\quad 0$

p(s) & q(s) Hurwitz

2. Residues at poles on imaginary axis must be real and positive.

$q(s) = s^2 + s + 4$

$= \left(s + \frac{1 + \sqrt{15}j}{2}\right)\left(s + \frac{1 - \sqrt{15}j}{2}\right)$

$\therefore s = \frac{-1 \pm \sqrt{15}j}{2}$

no poles at imaginary axis . condition satisfied

3. $A(\omega^2) \geq 0 \cdot \quad 0 < \omega < \infty$

$A(\omega^2) = m_1 m_2 - n_1 n_2 (s) \big|_{s=j\omega}$

$= \left(s^2 + \frac{3}{4}\right)(s^2 + 4) - \frac{3}{4}s \times s \big|_{s=j\omega}$

$= s^4 + \frac{3s^2}{4} + 4s^2 + 3 - \frac{3}{4}s^2 \big|_{s=j\omega}$

$A(\omega^2) = s^4 + 4s^2 + 3 = (j\omega)^4 + 4(j\omega)^2 + 3$

$= \omega^4 - 4\omega^2 + 3$

At $\omega = 1.5$, $A(\omega^2) = 1 - 4 + 3 < 0$ condition not satisfied

function is not PRF.

14. $F(s) = \frac{2s^2+2s+4}{(s+1)(s^2+2)}$

solution:

$$F(s) = \frac{2s^2+2s+4}{s^3+s^2+2s+2}$$

1. p(s) & q(s) must be Hurwitz

$p(s) = 2s^2+2s+4$

$q(s) = s^3+s^2+2s+2$

$2s\,)\,2s^2+4\,(\,s$
$\underline{2s^2}$
$\cancel{0 \; 2s \; t}$
4

$s^2+2\,)\,s^3+2s\,(\,s$
$\underline{s^3+2s}$
$\underline{0}$

Terminates abruptly

$\therefore q'(s) = 2s$

p(s) & q(s) Hurwitz

2. Residues of poles on imaginary axis must be real and positive

$q(s) = (s+1)(s^2+2) = (s+1)(s-\sqrt{2}j)(s+\sqrt{2}j)$

There are poles on imaginary poles. Hence find out its residues.

$$F(s) = \frac{2s^2+2s+4}{(s+1)(s^2+2)} = \frac{K_1}{s+1} + \frac{K_2}{s+\sqrt{2}j} + \frac{K_2^*}{s-\sqrt{2}j}$$

To find K_1, put $s = -1$

$$2(-1)^2+2(-1)+4 = K_1((-1)^2+2)$$

$$K_1 = \frac{4}{3}$$

To find K_2, put $s = -\sqrt{2}j$

$$2(-\sqrt{2}j)+2(-\sqrt{2}j)+4 = K_2(-\sqrt{2}j+1)(-2\sqrt{2}j)$$

$$K_2 = \frac{1}{1-\sqrt{2}j}$$

$$K_2^* = \left(\frac{1}{1-\sqrt{2}j}\right)^* = \frac{1}{1+\sqrt{2}j}$$

Since the residues K_2 & K_2^* are not real

$F(s)$ is not PRF.

15. which of the following are LC immittances?

a) $\dfrac{s(s^2+1)(s^2+9)}{(s^2+4)(s^2+16)}$ (b) $\dfrac{(s^2+1)(s^2+9)}{s(s^2+4)}$

c) $\dfrac{s(s^2+9)}{(s^2+1)(s^2+16)}$ d) $\dfrac{s^4+3s^2+2}{s^3+3s}$

Solution:

a) $H(s) = \dfrac{s(s^2+1)(s^2+9)}{(s^2+4)(s^2+16)}$

i) poles $\Rightarrow (s^2+4)=0$ $(s^2+16)=0$

$\Rightarrow s = \pm j2$ $s = \pm j4$

Zeros $\Rightarrow s=0$, $(s^2+1)=0$ $(s^2+9)=0$

$s =$ $s =$

ii) poles and zeros are interlacing

iii) Zero $(s=0)$ at the origin

iv) pole at infinity

v) Residues of poles:

$$\frac{s(s^2+1)(s^2+9)}{(s^2+4)(s^2+16)} = \frac{A}{(s+2j)} + \frac{B}{(s+4j)} + \frac{A^*}{(s-2j)} + \frac{B}{(s-4j)}$$

At $s = -2j$,

$$A = \left.\frac{(s+2j)\, s(s^2+1)(s^2+9)}{(s+2j)(s-2j)(s^2+16)}\right|_{s=-2j}$$

$$= \frac{(+2j)(-4+1)(-4+9)}{(-4j)(-4+16)} = \frac{(-2j)\times -15}{-4j \times 12} = \frac{-15}{24}$$

Since the residues at this pole is not positive real. $H(s)$ is not an LC immitance.

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b) $H(s) = \dfrac{(s^2+1)(s^2+9)}{s(s^2+4)}$

i) poles $\Rightarrow s=0,\ s=\pm j2$

zeros $\Rightarrow s=\quad ,\ s=$

poles & zeros are simple. They lie on jω axis

ii) poles and zeros are alternate

iii) pole at $s=0$ & zero at $s=\infty$

iv) Residues of poles:

$$\frac{(s^2+1)(s^2+9)}{s(s^2+4)} = \frac{A}{s} + \frac{B}{(s+j2)} + \frac{B^*}{(s-j2)}$$

$$A = \cancel{s}\,\frac{(s^2+1)(s^2+9)}{\cancel{s}(s^2+4)}\Bigg|_{s=0} = \frac{1\times 9}{4} = \frac{9}{4}$$

$$B = \frac{\cancel{(s+2j)}(s^2+1)(s^2+9)}{s\,\cancel{(s+2j)}(s-2j)}\Bigg|_{s=-2j}$$

$$= \frac{(-4+1)(-4+9)}{(-2j)\times(-4j)} = \frac{15}{8}$$

$$B^* = \frac{\cancel{(s-2j)}(s^2+1)(s^2+9)}{s(s+2j)\cancel{(s-2j)}}\Bigg|_{s=2j} = \frac{(-4+1)(-4+9)}{(-j2)(-j4)} = \frac{15}{8}$$

All the residues are of the poles are positive and real.

Hence it is an LC immitance

c) $H(s) = \dfrac{s(s^2+9)}{(s^2+1)(s^2+16)}$

1. poles $\Rightarrow (s^2+1) \Rightarrow s = \pm j$

$(s^2+16) \Rightarrow s = \pm 4j$

Zeros $\Rightarrow s=0$, $(s^2+9)=0 \Rightarrow s=$

poles and zeros are simple and lie on the $j\omega$ axis

2. poles and zeros are Interlacing

3. zeros at $s=0$ & poles at $s=\infty$

4. Residues of poles,

$$\frac{s(s^2+9)}{(s^2+1)(s^2+16)} = \frac{A}{(s+j)} + \frac{A^*}{(s-j)} + \frac{B}{(s+4j)} + \frac{B^*}{(s-4j)}$$

$$A = (s+j)\frac{s(s^2+9)}{(s+j)(s-j)(s^2+16)}\Bigg|_{s=-j} \Rightarrow \frac{-j(-1+9)}{-2j\times(-1+16)} = \frac{4}{15}$$

$$A^* = \frac{4}{15}$$

$$B = (s+4j)\frac{s(s^2+9)}{(s^2+1)(s+4j)(s-4j)}\Bigg|_{s=-4j} = \frac{(-4j)(-16+9)}{(-16+1)(-j8)} = \frac{7}{30}$$

$$B^* = \frac{7}{30}$$

Thus all the residues are real and positive

Hence it is an LC immitance.

(d) $H(s) = \dfrac{s^4+3s^2+2}{s^3+3s} = \dfrac{s^4+3s^2+2}{s(s^2+3)}$

i) poles are at $s=0$, $s=\pm j\sqrt{3}$

ii) Residues,

$$\frac{s^4+3s^2+2}{s(s^2+3)} = \frac{A}{s} + \frac{B}{s+j\sqrt{3}} + \frac{B^*}{s-j\sqrt{3}}$$

$$A = \cancel{s}\,\frac{s^4+3s^2+2}{\cancel{s}(s^2+3)}\bigg|_{s=0} = \frac{2}{3}$$

$$B = \cancel{(s+j\sqrt{3})}\,\frac{s^4+3s^2+2}{s\,\cancel{(s+j\sqrt{3})}(s-j\sqrt{3})}\bigg|_{s=-j\sqrt{3}}$$

$$= \frac{9-9+2}{(-j\sqrt{3})(-2j\sqrt{3})} = \frac{2}{-2\times 3} = \frac{2}{-6} = \frac{-1}{3}$$

Since the residues is not positive and real, $H(s)$ is not an LC immitance.

16. obtain Cauer I & Cauer II realization for LC function in (c) (1 problem)

$$H(s) = \frac{s(s^2+9)}{(s^2+1)(s^2+16)}$$

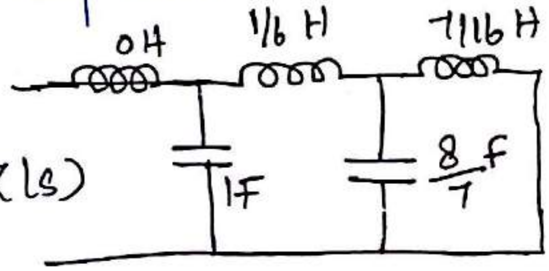
Solution:

a) Cauer I form:

$$H(s) = \frac{s^3+9s}{s^4+s^2+16s^2+16} = \frac{1}{\dfrac{s^4+\cancel{16}17s^2+16}{s^3+9s}}$$

$$s^3+9s \,\big)\, s^4+17s^2+16 \,\big(\, s \rightarrow C_1$$
$$s^4+9s^2$$
$$8s^2+16 \,\big)\, s^3+9s \,\big(\, \frac{s}{8} \rightarrow L_2$$
$$s^3+2s$$
$$7s \,\big)\, 8s^2+16 \,\big(\, \frac{8s}{7} - C_3$$
$$8s^2$$
$$16 \,\big)\, 7s \,\big(\, \frac{7s}{16} - L_4$$
$$7s$$
$$0$$

Zero $\Rightarrow$ s^4+17s^2+16 $\Rightarrow$ zero at $s=\infty$ so, shunt capacitor



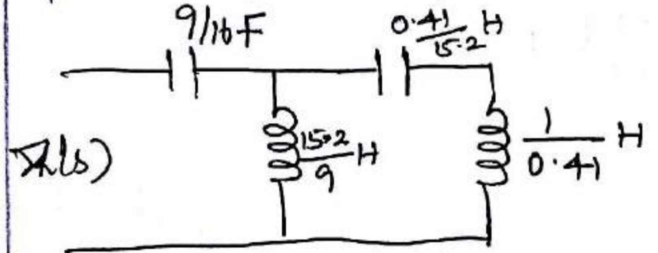
$$Z(s) = 0 + \cfrac{1}{s + \cfrac{1}{\left(\frac{s}{8}\right) + \cfrac{1}{\left(\frac{8s}{7}\right) + \cfrac{1}{\left(\frac{7s}{16}\right)}}}}$$

b) Cauer II form:

$$H(s) = \frac{s^3+9s}{s^4+17s^2+16} = \frac{1}{\dfrac{16+17s^2+s^4}{9s+s^3}}$$

$$9s+s^3\overline{)\,16+17s^2+s^4\,}\left(\frac{16}{9s}\right.$$
$$16+1.8s^2$$
$$\overline{\qquad\qquad\qquad}$$
$$15.2s^2+s^4\overline{)\,9s+s^3\,}\left(\frac{9}{15.2s}\right.$$
$$9s+0.59s^3$$
$$\overline{\qquad\qquad\qquad}$$
$$0.41s^3\overline{)\,15.2s^2+s^4\,}\left(\frac{15.2}{0.41s}\right.$$
$$15.2s^2$$
$$\overline{\qquad\qquad\qquad}$$
$$s^4\overline{)\,0.41s^3\,}\left(\frac{0.41}{s}\right.$$
$$0.41s^3$$
$$\overline{\qquad\qquad}$$
$$0$$

pole $\Rightarrow 9s+s^3 \Rightarrow 0$ at $s=0$, so series capacitor



$$Z(s)=\cfrac{1}{\left(\frac{16}{9s}\right)+\cfrac{1}{\left(\frac{9}{15.2s}\right)+\cfrac{1}{\left(\frac{15.2}{0.41s}\right)+\cfrac{1}{\left(\frac{0.41}{s}\right)}}}}$$

17. Obtain foster I and foster II for the function

$$Z(s)=\frac{(s^2+1)\,(s^2+9)}{s\,(s^2+4)}$$

Solution:

The given function satisfy the properties of LC function.

First Foster form:

$$Z(s) = \frac{(s^2+1)(s^2+9)}{s(s^2+4)}$$

By partial fraction,

$$Z(s) = \frac{A}{s} + \frac{B}{s+2j} + \frac{B^*}{s-2j}$$

$$A = \cancel{s}\, \frac{(s^2+1)(s^2+9)}{\cancel{s}(s^2+4)}\Bigg|_{s=0} = \frac{9}{4}$$

$$B = \cancel{(s+2j)}\, \frac{(s^2+1)(s^2+9)}{s\cancel{(s+2j)}(s-2j)}\Bigg|_{s=-2j} = \frac{-3\times 5}{(-2j)(-4j)} = \frac{-15}{-8}$$

$$= \frac{15}{8}$$

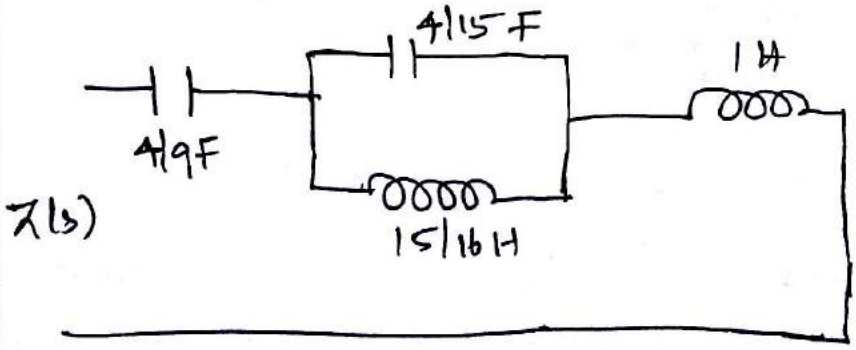
$$\therefore Z(s) = \frac{(9/4)}{s} + \frac{2\,(15/8)\,s}{s^2+4}$$

$$\therefore C_0 = \frac{1}{p_0} = \frac{1}{(9/4)} = \frac{4}{9}F$$

By inspection, $L_\infty = H = 1$

$$C_1 = \frac{1}{2p_1} = \frac{8}{2\times 15} = \frac{4}{15}F$$

$$L_1 = \frac{2p_1}{\omega_2^2} = \frac{2\times 15}{8\times 4} = \frac{15}{16}H$$



Second foster form:

$$Y(s) = \frac{s(s^2+4)}{(s^2+1)(s^2+9)}$$

By partial fraction,

$$Y(s) = \frac{A}{s+j} + \frac{A^*}{s-j} + \frac{B}{s+3j} + \frac{B^*}{s-3j}$$

$$A = \cancel{(s+j)} \frac{s(s^2+4)}{\cancel{(s+j)}(s-j)(s^2+9)}\Bigg|_{s=-j} = \frac{-j \times 3}{-2j \times 8} = \frac{3}{16}$$

$$B = \cancel{(s+3j)} \frac{s(s^2+4)}{(s^2+1)\cancel{(s+3j)}(s-3j)}\Bigg|_{s=-3j} = \frac{-3j \times -5}{-8 \times -6j} = \frac{15}{16}$$

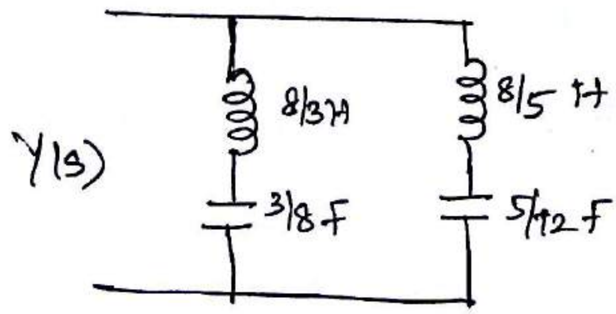
$$Y(s) = \frac{2(3/16)s}{s^2+1} + \frac{2(5/16)s}{s^2+9}$$

$$L_2 = \frac{1}{2P_2} = \frac{16}{2 \times 3} = \frac{8}{3}\,H$$

$$C_2 = \frac{2P_2}{\omega_2^2} = \frac{2 \times 3}{16 \times 1} = \frac{3}{8}\,F$$

$$L_4 = \frac{1}{2P_4} = \frac{16}{2 \times 5} = \frac{8}{5}\,H$$

$$C_4 = \frac{2P_4}{\omega_4^2} = \frac{2 \times 5}{16 \times 9} = \frac{5}{72}\,F$$



18. Obtain Cauer I & II for the function

$$Z(s) = \frac{s^4+10s^2+9}{s^3+4s}$$

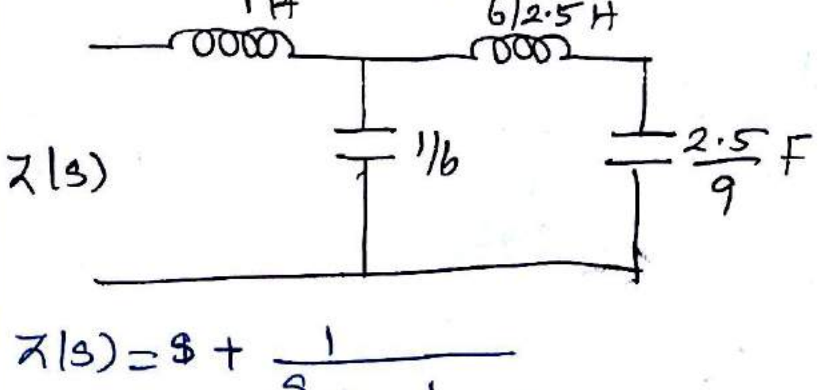
Solution:

It is an LC function

Cauer I :

$$s^3+4s \;)\; s^4+10s^2+9 \;(s$$
$$s^4+4s^2$$
$$\overline{6s^2+9 \;)\; s^3+4s \;(s/6}$$
$$s^3+1.5s$$
$$\overline{2.5s \;)\; 6s^2+9 \;(\frac{6s}{2.5}}$$
$$6s^2$$
$$\overline{9 \;)\; 2.5s \;(\frac{2.5s}{9}}$$
$$2.5s$$
$$\overline{0}$$

pole at $s=\infty$, so series Inductor (1st Element)
pole at $s=0$, so shunt Capacitor (last Element)



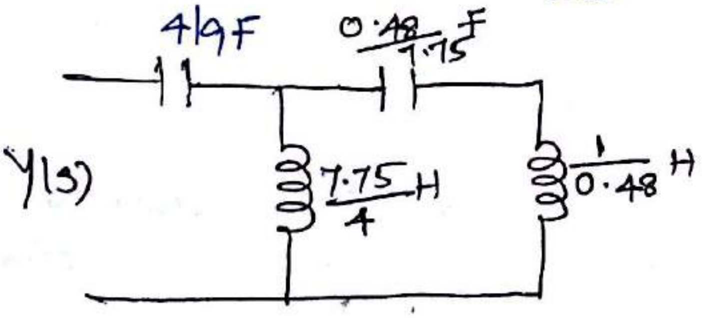
$$Z(s) = s + \cfrac{1}{\cfrac{s}{6} + \cfrac{1}{\cfrac{6s}{2.5} + \cfrac{1}{\cfrac{2.5s}{9}}}}$$

Cauer II :

$$Y(s) = \frac{9+10s^2+s^4}{4s+s^3}$$

$$4s+s^3 \,)\, 9+10s^2+s^4 \,(\frac{9}{4s}$$
$$9+2.25s^2$$
$$7.75s^2+s^4 \,)\, 4s+s^3 \,(\frac{4}{7.75s}$$
$$4s+0.52s^3$$
$$0.48s^3 \,)\, 7.75s^2+s^4 \,(\frac{7.75}{0.48s}$$
$$7.75s^2$$
$$s^4 \,)\, 0.48s^3 \,(\frac{0.48}{s}$$
$$0.48s^3$$
$$0$$

pole at $s=0$ & $s=\alpha$ so, 1st element = Capacitor
Last " = Inductor



$$Y(s) = \frac{9}{4s} + \cfrac{1}{\cfrac{4}{7.75s} + \cfrac{1}{\cfrac{7.75}{0.48s} + \cfrac{1}{\cfrac{0.48}{s}}}}$$

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19. Obtain Foster and Cauer realization of LC function

$$Z(s) = \frac{6s^4 + 42s^2 + 48}{s^5 + 18s^3 + 48s}$$

Solution:

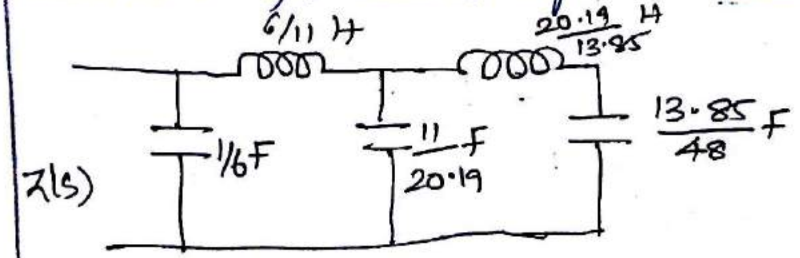
For obtaining cauer form, num < deno so inverse.

$$Z(s) = \frac{1}{\frac{s^5 + 18s^3 + 48s}{6s^4 + 42s^2 + 48}}$$

Cauer I:

$$
\begin{array}{l}
6s^4+42s^2+48)\ s^5+18s^3+48s\ (\frac{s}{6} \\
\qquad\qquad\qquad s^5+7s^3+8s \\
\hline
11s^3+40s)\ 6s^4+42s^2+48\ (\frac{6s}{11} \\
\qquad\qquad 6s^4+21.81s^2 \\
\hline
20.19s^2+48)\ 11s^3+40s\ (\frac{11s}{20.19} \\
\qquad\qquad 11s^3+26.15s \\
\hline
13.85s)\ 20.19s^2+48\ (\frac{20.19s}{13.85} \\
\qquad\qquad 20.19s^2 \\
\hline
48)\ 13.85s\ (\frac{13.85s}{48} \\
\qquad 13.85s \\
\hline
\qquad 0
\end{array}
$$

Zero at $s=\infty$, 1^{st} element - cap



$$Z(s) = \cfrac{1}{\cfrac{s}{6} + \cfrac{1}{\cfrac{6s}{11} + \cfrac{1}{\cfrac{11s}{20.19} + \cfrac{1}{\cfrac{20.19s}{13.85} + \cfrac{1}{\cfrac{13.85s}{48}}}}}}$$

Case II:

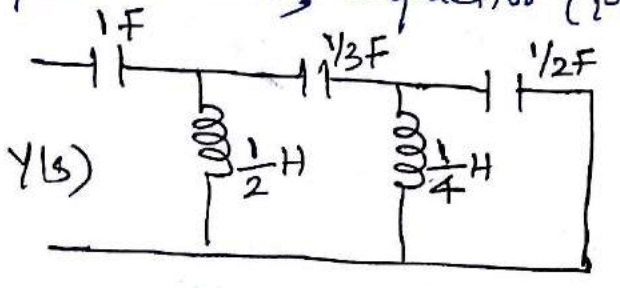
$$Y(s) = \frac{1}{\frac{48s+18s^3+s^5}{48+42s^2+6s^4}}$$

$$48+42s^2+6s^4\,)\;48s+18s^3+s^5\;(s$$
$$48s+42s^3+6s^5$$
$$\overline{-24s^3-5s^5}$$

Since the values are negative,

$$48s+18s^3+s^5\,)\;48+42s^2+6s^4\;(\tfrac{1}{s}$$
$$48+18s^2+s^4$$
$$24s^2+5s^4\,)\;48s+18s^3+s^5\;(\tfrac{48}{24s}$$
$$48s+10s^3$$
$$8s^3+s^5\,)\;24s^2+5s^4\;(\tfrac{24}{8s}$$
$$24s^2+3s^4$$
$$2s^4\,)\;8s^3+s^5\;(\tfrac{8}{2s}$$
$$8s^3$$
$$s^5\,)\;2s^4\;(\tfrac{2}{s}$$
$$2s^4$$
$$\overline{0}$$

pole at $s=0$, capacitor (1st Element)



$$Y(s) = \frac{1}{s} + \cfrac{1}{\frac{2}{s} + \cfrac{1}{\frac{3}{s} + \cfrac{1}{\frac{4}{s} + \cfrac{1}{\frac{2}{s}}}}}$$

Foster I form:

$$Z(s) = \frac{6s^4+42s^2+48}{s(s^4+18s^2+48)}$$

The roots of s^4+18s^2+48 are

$$s_{1,2}^2 = \frac{-b \pm \sqrt{b^2-4ac}}{2a} = \frac{-18 \pm \sqrt{18^2-4\times1\times48}}{2}$$

$$s_1^2 = -14.75 \ , \ s_2^2 = -3.26$$

$$Z(s) = \frac{6s^4+42s^2+48}{s(s^2+14.75)(s^2+3.26)}$$

$$Z(s) = \frac{A}{s} + \frac{B}{s+3.84j} + \frac{B^*}{s-3.84j} + \frac{C}{s+1.8j} + \frac{C^*}{s-1.8j}$$

$$A = \cancel{s}\frac{(6s^4+42s^2+48)}{\cancel{s}(s^4+18s^2+48)}\bigg|_{s=0} \Rightarrow \frac{48}{48} = 1$$

$$B = \cancel{(s+3.84j)}\frac{6s^4+42s^2+48}{s\cancel{(s+3.84j)}(s-3.84j)(s^2+3.26)}\bigg|_{s=-3.84j}$$

$$= \frac{1304.4 - 619.3 + 48}{(-3.84j)(-7.68j)(-11.49)} = \frac{733.1}{338.85} = 2.16$$

$$C = \cancel{(s+1.8j)}\frac{(6s^4+42s^2+48)}{s(s^2+14.75)\cancel{(s+1.8j)}(s-1.8j)}\bigg|_{s=-1.8j}$$

$$= \frac{62.99 - 136.08 + 48}{(-1.8j)(11.51)(-3.6j)} = \frac{-25.09}{-74.58} = 0.336$$

$$Z(s) = \frac{1}{s} + \frac{2(2.16)s}{s^2+14.75} + \frac{2(0.336)s}{s^2+3.26}$$

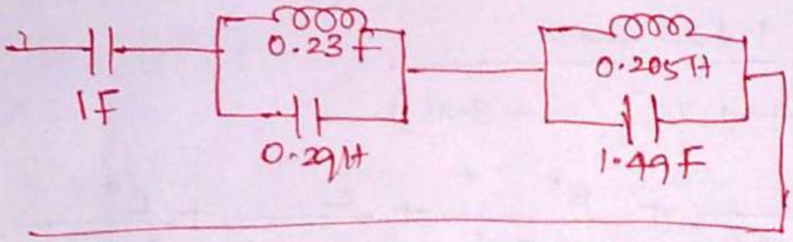
$$C_0 = \frac{1}{P_0} = \frac{1}{1} = 1 \ , \ L_\alpha = 0$$

$$C_1 = \frac{1}{2p_1} = \frac{1}{2\times 2.16} = 0.23$$

$$L_1 = \frac{2p_1}{\omega_1^2} = \frac{2\times 2.16}{14.75} = 0.29$$

$$C_2 = \frac{1}{2p_2} = \frac{1}{2\times 0.336} = 1.488$$

$$L_2 = \frac{2p_2}{\omega_2^2} = \frac{2\times 0.336}{3.26} = 0.205$$



Foster II form:

$$Z(s) = \frac{s(s^4+18s^2+48)}{6s^4+42s^2+48}$$

The roots of $6s^4+42s^2+48$ is

$$s^2_{1,2} = \frac{-b \pm \sqrt{b^2-4ac}}{2a} = \frac{-42 \pm \sqrt{42^2 - 4\times 6\times 48}}{2\times 6} = \frac{-42 \pm \sqrt{612}}{12}$$

$$s^2_1 = -1.44 \ , \ s^2_2 = -5.56$$

$$\therefore Z(s) = \frac{s(s^4+18s^2+48)}{(s^2+1.44)(s^2+5.56)}$$

$$Z(s) = \frac{A}{s+1.20j} + \frac{A^*}{s-1.20j} + \frac{B}{s+2.36j} + \frac{B^*}{s-2.36j}$$

$$A = \cancel{(s+1.2j)} \frac{s(s^4+18s^2+48)}{\cancel{(s+1.2j)}(s-1.2j)(s^2+5.56)}\Bigg|_{s=-1.2j}$$

$$= \frac{-1.2j\,(2.07-25.92+48)}{(-2.4j)(4.12)} = \frac{28.98}{9.9} = 2.93$$

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$$B = \cancel{(s+2.36j)} \frac{s(s^4+18s^2+48)}{(s^2+1.44)\cancel{(s+2.36j)}(s-2.36j)} \Bigg|_{s=-2.36j}$$

$$= \frac{-2.36j(31.02 - 100.26 + 48)}{(-4.13)(-4.72j)}$$

$$= \frac{50.13j}{19.49j}$$

$$B = 2.57$$

$$Y(s) = \frac{2(2.93)s}{s^2+1.44} + \frac{2(2.57)s}{s^2+5.56}$$

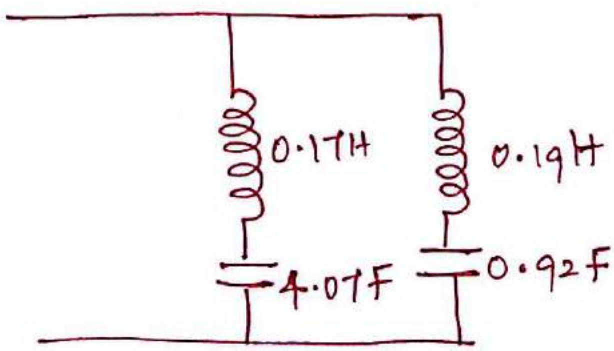
$L_0 = 0$ & $C_\infty = 0$

$$L_1 = \frac{1}{2p_1} = \frac{1}{2\times 2.93} = 0.17$$

$$C_1 = \frac{2p_1}{\omega_1^2} = \frac{2\times 2.93}{1.44} = 4.07$$

$$L_2 = \frac{1}{2p_2} = \frac{1}{2\times 2.57} = 0.19$$

$$C_2 = \frac{2p_2}{\omega_2^2} = \frac{2\times 2.57}{5.56} = 0.92$$



obtain cauer I form for the LC function

20) $Z(s) = \dfrac{s^3+2s}{s^4+4s^2+3}$

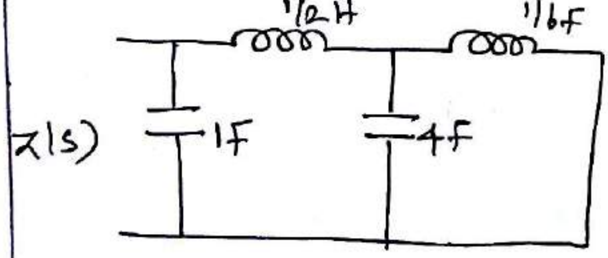
Solution:

num power is < Deno power so inverse.

$$Z(s) = \frac{1}{\dfrac{s^4+4s^2+3}{s^3+2s}}$$

$$\begin{array}{l}
s^3+2s \;)\; s^4+4s^2+3 \;(\; s \\
\qquad\qquad s^4+2s^2 \\
\hline
2s^2+3 \;)\; s^3+2s \;(\; \frac{s}{2} \\
\qquad\qquad s^3+\frac{3s}{2} \\
\hline
\frac{s}{2} \;)\; 2s^2+3 \;(\; 4s \\
\qquad\qquad 2s^2 \\
\hline
3 \;)\; \frac{s}{2} \;(\; \frac{s}{6} \\
\qquad\qquad \frac{s}{2} \\
\hline
\qquad\qquad 0
\end{array}$$

Zero at $s=\infty$, 1st element capacitor



$$Z(s) = \cfrac{1}{s+\cfrac{1}{\cfrac{s}{2}+\cfrac{1}{4s+\cfrac{1}{\cfrac{s}{6}}}}}$$

21) obtain foster I & II for the function

$$Z(s) = \frac{s(s^2+4)}{(s^2+2)(s^2+8)}$$

<u>Solution:</u>

It is an LC function

<u>Foster I form:</u>

By partial fraction,

$$Z(s) = \frac{A}{s+\sqrt{2}j} + \frac{A^*}{s-\sqrt{2}j} + \frac{B}{s+2\sqrt{2}j} + \frac{B^*}{s-2\sqrt{2}j}$$

$$A = \cancel{(s+\sqrt{2}j)} \left. \frac{s(s^2+4)}{\cancel{(s+\sqrt{2}j)}(s-\sqrt{2}j)(s^2+8)} \right|_{s=-\sqrt{2}j}$$

$$= \frac{-\cancel{\sqrt{2}j} \times \cancel{2}}{-\cancel{2}\cancel{\sqrt{2}j} \times 6} = \frac{1}{6}$$

$$B = \cancel{(s+2\sqrt{2}j)} \left. \frac{s(s^2+4)}{(s^2+2)\cancel{(s+2\sqrt{2}j)}(s-2\sqrt{2}j)} \right|_{s=-2\sqrt{2}j}$$

$$= \frac{-\cancel{2}\cancel{\sqrt{2}j} \times -\cancel{4}\,2}{-6 \times -\cancel{4}_{2}\cancel{\sqrt{2}j}} = \frac{1}{3}$$

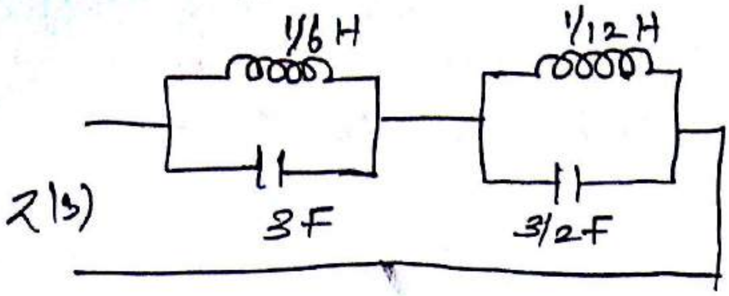
$$\therefore Z(s) = \frac{2(1/6)s}{s^2+2} + \frac{2(1/3)s}{s^2+8}$$

$$C_1 = \frac{1}{2p_1} = \frac{6}{2\times 1} = 3$$

$$L_1 = \frac{2p_1}{\omega_1^2} = \frac{2\times 1}{6\times 2} = \frac{1}{6}$$

$$C_2 = \frac{1}{2p_2} = \frac{3}{2\times 1} = \frac{3}{2}$$

$$L_2 = \frac{2p_2}{\omega_2^2} = \frac{2\times 1}{3\times 8} = \frac{1}{12}$$



Foster II form:

$$Y(s) = \frac{(s^2+2)\,(s^2+8)}{s\,(s^2+4)}$$

By partial fraction,

$$Y(s) = \frac{A}{s} + \frac{B}{s+2j} + \frac{B^*}{s-2j}$$

$$A = \cancel{s}\,\frac{(s^2+2)\,(s^2+8)}{\cancel{s}\,(s^2+4)}\Bigg|_{s=0} = \frac{16}{4} = 4$$

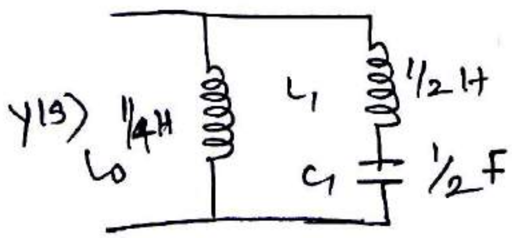
$$B = (\cancel{s+2j})\,\frac{(s^2+2)\,(s^2+8)}{s\,(\cancel{s+2j})(s-2j)}\Bigg|_{s=-2j}$$

$$= \frac{-2 \times -4}{-2j \times -4j} = +1$$

$$\therefore Z(s) = \frac{4}{s} + \frac{2\,(1)\,s}{s^2+4}$$

$$L_0 = \frac{1}{P_0} = \frac{1}{4}, \quad L_1 = \frac{1}{2P_1} = \frac{1}{2\times 1} = \frac{1}{2}$$

$$C_1 = \frac{2P_1}{\omega_1^2} = \frac{2\times 1}{4} = \frac{2}{4} = \frac{1}{2}$$



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22) For the following functions find whether it is an RC impedance function and synthesize in the four Canonical forms.

$$Z(s) = \frac{(s+1)(s+4)}{s(s+2)}$$

Solution:

1. A pole at $s=0$
2. All poles and zeros on real negative axis
3. Alternate poles and zeros
4. Residues are real and positive

$\therefore Z(s)$ is RC impedance function

Foster I form:

$$Z(s) = \frac{A}{s} + \frac{B}{s+2}$$

$$s^2+2s \,)\, s^2+5s+4 \,(\, 1$$
$$\underline{s^2+2s}$$
$$\underline{3s+4}$$

$$Z(s) = 1 + \frac{3s+4}{s(s+2)}$$

$$Z(s) = 1 + \frac{A}{s} + \frac{B}{s+2}$$

By heaviside method,

$$A = \cancel{s}\frac{3s+4}{\cancel{s}(s+2)}\Bigg|_{s=0} = \frac{4}{2} = 2$$

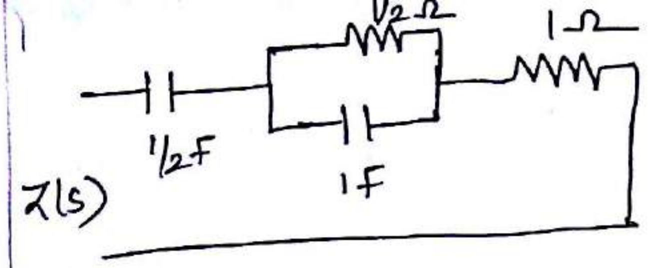
$$B = \cancel{(s+2)}\frac{3s+4}{s\cancel{(s+2)}}\Bigg|_{s=-2} = \frac{-2}{-2} = 1$$

$$\therefore Z(s) = 1 + \frac{2}{s} + \frac{1}{s+2}$$

$R_\infty = H = 1\,\Omega$

$$C_0 = \frac{1}{P_0} = \frac{1}{2}$$

$$C_1 = \frac{1}{P_1} = \frac{1}{1} = 1 \;,\; P_1 = \frac{1}{\sigma_1 C_1} = \frac{1}{2\times 1} = \frac{1}{2}$$



Foster II form:

$$Y(s) = \frac{s(s+2)}{(s+1)(s+4)}$$

$$= \frac{s^2+2s}{s^2+5s+4}$$

$$s^2+5s+4 \,\big)\, s^2+2s \,\big(\, 1$$
$$\underline{s^2+5s+4}$$
$$\underline{-3s-4}$$

$$Y(s) = 1 + \frac{(-3s-4)}{(s+1)(s+4)}$$

$$Y(s) = 1 + \frac{A}{s+1} + \frac{B}{s+4}$$

$$A = \cancel{(s+1)} \frac{(-3s-4)}{\cancel{(s+1)}(s+4)} \Bigg|_{s=-1} = \frac{-1}{3}$$

Since residues are negative, we have to expand $\frac{Y(s)}{s}$ as follows,

$$\frac{Y(s)}{s} = \frac{\cancel{s}(s+2)}{\cancel{s}(s+1)(s+4)} = \frac{s+2}{(s+1)(s+4)}$$

$$\frac{Y(s)}{s} = \frac{A}{s+1} + \frac{B}{s+4}$$

$$A = \cancel{(s+1)} \frac{s+2}{\cancel{(s+1)}(s+4)} \Bigg|_{s=-1} = \frac{1}{3}$$

$$B = \cancel{(s+4)} \frac{s+2}{(s+1)\cancel{(s+4)}} \Bigg|_{s=-4} = \frac{-2}{-3} = \frac{2}{3}$$

$$\frac{Y(s)}{s} = \frac{(1/3)}{s+1} + \frac{(2/3)}{s+4}$$

$$Y(s) = \frac{(1/3)s}{s+1} + \frac{(2/3)s}{s+4}$$

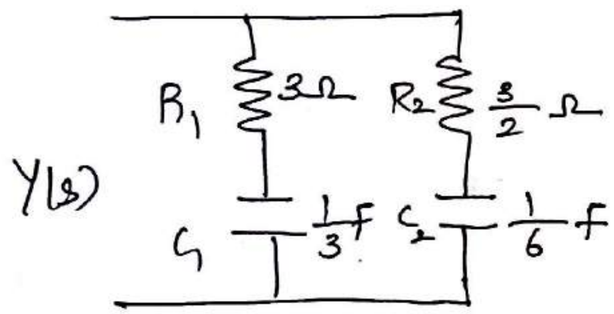
$\therefore\ R_0 = 0$ & $C_\infty = 0$

$$\frac{P_1 s}{s+\sigma_1} \Rightarrow \frac{(1/3)s}{s+1} \Rightarrow P_1 = \frac{1}{3} \Rightarrow R_1 = \frac{1}{P_1} = \frac{1}{1/3} = 3$$

$$C_1 = \frac{1}{\sigma_1 R_1} = \frac{1}{1\times 3} = \frac{1}{3}$$

$$\frac{P_2 s}{s+\sigma_2} \Rightarrow \frac{(2/3)s}{s+4} \Rightarrow P_2 = \frac{2}{3} \Rightarrow R_2 = \frac{1}{P_2} = \frac{1}{(2/3)} = \frac{3}{2}$$

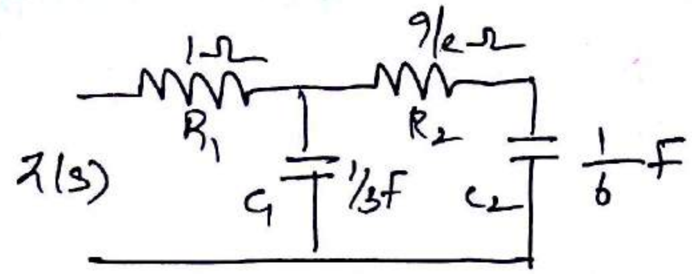
$$C_2 = \frac{1}{\sigma_2 R_2} = \frac{1}{4\times(3/2)} = \frac{2}{4\times 3} = \frac{1}{6}$$



Cauer I form:

$$Z(s) = \frac{s^2+5s+4}{s^2+2s}$$

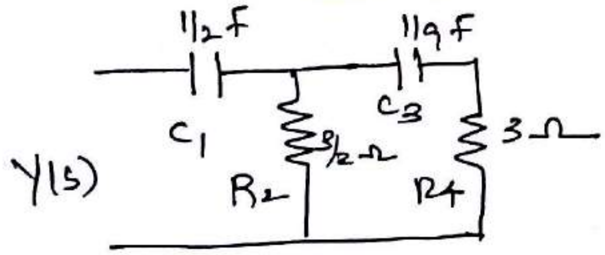
$$
\begin{array}{l}
s^2+2s\ )\ s^2+5s+4\ (\ 1 \leftarrow R_1 \\
\qquad\qquad\ \ s^2+2s \\
\hline
3s+4\ )\ s^2+2s\ (\ \frac{s}{3} \leftarrow C_2 s \\
\qquad\qquad s^2+\frac{4s}{3} \\
\hline
\frac{2}{3}s\ )\ 3s+4\ (\ \frac{9}{2} \leftarrow R_2 \\
\qquad\qquad 3s \\
\hline
4\ )\ \frac{2}{3}s\ (\ \frac{1}{6}s \leftarrow C_4 s \\
\qquad\quad \frac{2}{3}s \\
\hline
\qquad\quad 0 \\
\hline
\end{array}
$$



Cauer II form:

$$Z(s) = \frac{4+5s+s^2}{2s+s^2}$$

$$\begin{array}{l} 2s+s^2 \overline{)\,4+5s+s^2\,}\left(\frac{2}{s} \leftarrow \frac{1}{C_1 s} \qquad C_1 = \frac{1}{2} \right. \\ \quad 4+2s \\ \hline 3s+s^2 \overline{)\,2s+s^2\,}\left(\frac{2}{3} \leftarrow \frac{1}{R_2} \qquad R_2 = \frac{3}{2} \right. \\ \quad 2s+\frac{2s^2}{3} \\ \hline \frac{s^2}{3} \overline{)\,3s+s^2\,}\left(\frac{9}{s} \leftarrow \frac{1}{C_3 s} \qquad C_3 = \frac{1}{9} \right. \\ \quad 3s \\ \hline s^2 \overline{)\,\frac{s^2}{3}\,}\left(\frac{1}{3} \leftarrow \frac{1}{R_4} \qquad R_4 = 3 \right. \\ \quad \frac{s^2}{3} \\ \hline \quad 0 \\ \hline \end{array}$$



23) Find whether the following is an RC impedance function and synthesize in the four canonical forms.

$$Z(s) = \frac{s(s+1)(s+4)}{(s+3)}$$

Solution:

1. Alternate poles and zeros
2. All poles and zeros on real negative axis
3. no pole at $s=0$

$\therefore$ It is not an RC impedance function

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24) synthesize the function in four canonical form.

$$Z(s) = \frac{(s+1)(s+4)(s+8)}{s(s+2)(s+6)}$$

solution:

Foster I form:

$$Z(s) = \frac{s^3+13s^2+44s+32}{s^3+8s^2+12s}$$

$$s^3+8s^2+12s \overline{) \; s^3+13s^2+44s+32 \;} (1$$
$$\underline{s^3+8s^2+12s}$$
$$5s^2+32s+32$$

$$Z(s) = 1 + \frac{5s^2+32s+32}{s(s+2)(s+6)}$$

$$= 1 + \frac{A}{s} + \frac{B}{s+2} + \frac{C}{s+6}$$

$$A = \cancel{s}\frac{(5s^2+32s+32)}{\cancel{s}(s+2)(s+6)}\Bigg|_{s=0} \Rightarrow \frac{32}{12} = \frac{8}{3}$$

$$B = \cancel{(s+2)}\frac{(5s^2+32s+32)}{s\cancel{(s+2)}(s+6)}\Bigg|_{s=-2} \Rightarrow \frac{20-64+32}{(-2)\times(4)} \Rightarrow \frac{-12}{-8}$$

$$= \frac{3}{2}$$

$$C = \cancel{(s+6)}\frac{(5s^2+32s+32)}{s(s+2)\cancel{(s+6)}}\Bigg|_{s=-6} \Rightarrow \frac{180-192+32}{-6\times-4} = \frac{20}{24}$$

$$= \frac{5}{6}$$

$$\therefore Z(s) = 1 + \frac{(8/3)}{s} + \frac{(3/2)}{s+2} + \frac{(5/6)}{s+6}$$

$$R_\alpha = H = 1\Omega$$

$$C_0 = \frac{1}{P_0} = \frac{1}{(8/3)} = \frac{3}{8}$$

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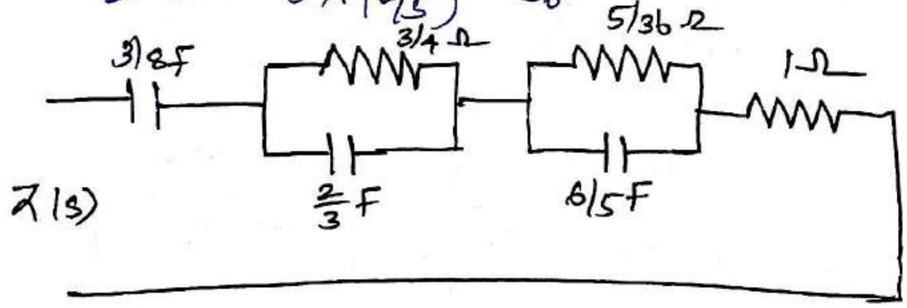
$$c_1 = \frac{1}{p_1} = \frac{1}{(3/2)} = \frac{2}{3}$$

$$p_1 = \frac{1}{\sigma_1 c_1} = \frac{1}{2 \times (2/3)} = \frac{3}{4}$$

$$c_2 = \frac{1}{p_2} = \frac{1}{(5/6)} = \frac{6}{5}$$

$$p_2 = \frac{1}{\sigma_2 c_2} = \frac{1}{6 \times (6/5)} = \frac{5}{36}$$



Foster II form:

$$Y(s) = \frac{s^3 + 8s^2 + 12s}{s^3 + 13s^2 + 44s + 32}$$

$$s^3 + 13s^2 + 44s + 32 \,\big)\, s^3 + 8s^2 + 12s \,\big(\, 1$$

$$\frac{s^3 + 13s^2 + 44s + 32}{-5s^2 - 32s - 32}$$

$$Y(s) = 1 + \frac{(-5s^2 - 32s - 32)}{(s+1)(s+4)(s+8)}$$

$$Y(s) = 1 + \frac{A}{s+1} + \frac{B}{s+4} + \frac{C}{s+8}$$

$$A = \cancel{(s+1)} \frac{(-5s^2 - 32s - 32)}{\cancel{(s+1)}(s+4)(s+8)} \bigg|_{s=-1} \Rightarrow \frac{-5 + \cancel{32} - \cancel{32}}{3 \times 7} = \frac{-5}{21}$$

Since the residues are negative, find $\frac{Y(s)}{s}$

$$\frac{Y(s)}{s} = \frac{\cancel{s}(s+2)(s+6)}{\cancel{s}(s+1)(s+4)(s+8)}$$

$$\frac{Y(s)}{s} = \frac{(s+2)(s+6)}{(s+1)(s+4)(s+8)}$$

$$\frac{Y(s)}{s} = \frac{A}{s+1} + \frac{B}{s+4} + \frac{C}{s+8}$$

$$A = \cancel{(s+1)} \left. \frac{(s+2)(s+6)}{\cancel{(s+1)}(s+4)(s+8)} \right|_{s=-1} \Rightarrow \frac{5}{21}$$

$$B = \cancel{(s+4)} \left. \frac{(s+2)(s+6)}{(s+1)\cancel{(s+4)}(s+8)} \right|_{s=-4} \Rightarrow \frac{-2 \times 2}{-3 \times 4} = \frac{1}{3}$$

$$C = \cancel{(s+8)} \left. \frac{(s+2)(s+6)}{(s+1)(s+4)\cancel{(s+8)}} \right|_{s=-8} \Rightarrow \frac{-6 \times -2}{-7 \times -4} \Rightarrow \frac{3}{7}$$

$$\frac{Y(s)}{s} = \frac{(5/21)}{s+1} + \frac{(1/3)}{s+4} + \frac{(3/7)}{s+8}$$

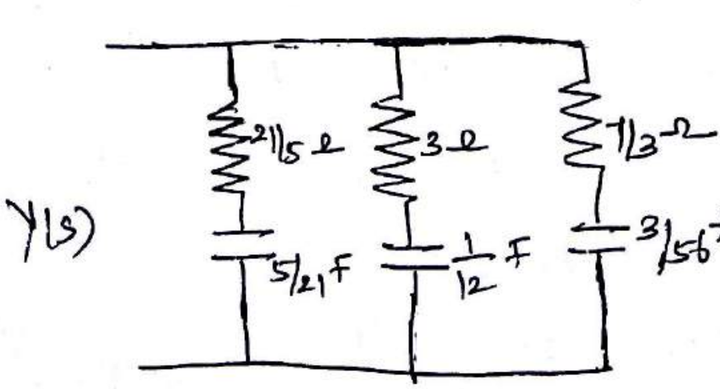
$$Y(s) = \frac{(5/21)s}{s+1} + \frac{(1/3)s}{s+4} + \frac{(3/7)s}{s+8}$$

$R_0 = 0$ & $C_\infty = 0$

$$P_1 \Rightarrow \frac{5}{21} \Rightarrow R_1 = \frac{1}{P_1} = \frac{1}{(5/21)} = \frac{21}{5},\ C_1 = \frac{1}{\sigma_1 R_1} = \frac{1}{1 \times (21/5)} = \frac{5}{21}$$

$$R_2 = \frac{1}{P_2} = \frac{1}{(1/3)} = 3,\ C_2 = \frac{1}{\sigma_2 R_2} = \frac{1}{4 \times 3} = \frac{1}{12}$$

$$R_3 = \frac{1}{P_3} = \frac{1}{(3/7)} = \frac{7}{3},\ C_3 = \frac{1}{\sigma_3 R_3} = \frac{1}{8 \times (7/3)} = \frac{3}{56}$$



Cauer I form:

$$s^3+8s^2+12s \,\big)\, s^3+13s^2+44s+32 \,\big(\, 1 \leftarrow R_1$$

$$s^3+8s^2+12s$$

$$5s^2+32s+32 \,\big)\, s^3+8s^2+12s \,\big(\, \frac{s}{5} \leftarrow C_2$$

$$s^3+\frac{32s^2}{5}+\frac{32s}{5}$$

$$\frac{8}{5}s^2+\frac{28}{5}s \,\big)\, 5s^2+32s+32 \,\big(\, \frac{25}{8} \leftarrow R_3$$

$$5s^2+\frac{35}{2}s$$

$$\frac{29}{2}s+32 \,\big)\, \frac{8}{5}s^2+\frac{28}{5}s \,\big(\, \frac{16s}{145} \leftarrow C_4$$

$$\frac{8}{5}s^2+\frac{512s}{145}$$

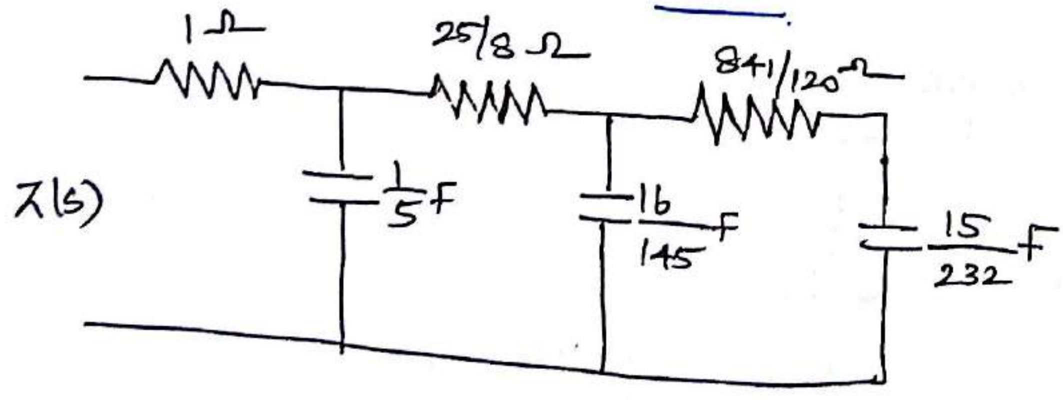
$$\frac{60}{29}s \,\big)\, \frac{29}{2}s+32 \,\big(\, \frac{841}{120} \leftarrow R_5$$

$$\frac{29}{2}s$$

$$32 \,\big)\, \frac{60s}{29} \,\big(\, \frac{15s}{232} \leftarrow C_5$$

$$\frac{60s}{29}$$

$$0$$



Cauer II form:

$$Z(s) = \frac{32 + 44s + 13s^2 + s^3}{12s + 8s^2 + s^3}$$

$$12s+8s^2+s^3 \,\big)\, 32+44s+13s^2+s^3 \,\big(\frac{8}{3s} \leftarrow \frac{1}{C_1 s} \Rightarrow C_1 = \frac{3}{8}$$
$$32 + \frac{64}{3}s + \frac{8}{3}s^2$$

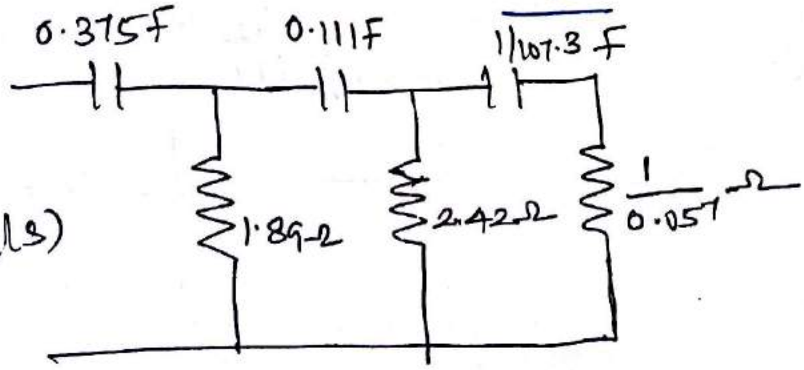
$$\frac{68}{3}s+\frac{31}{3}s^2+s^3 \,\big)\, 12s + 8s^2 + s^3 \,\big(\frac{9}{17} \leftarrow \frac{1}{R_1} \Rightarrow R_2 = \frac{17}{9}$$
$$12s + \frac{93}{17}s^2 + \frac{9}{17}s^3$$

$$\frac{43}{17}s^2+\frac{8}{17}s^3 \,\big)\, \frac{68}{3}s + \frac{31}{3}s^2 + s^3 \,\big(\frac{1156}{129s} \leftarrow \frac{1}{C_3 s} \Rightarrow C_3 = \frac{129}{1156}$$
$$\frac{68}{3}s + \frac{544}{129}s^2$$

$$\frac{2367}{387}s^2+s^3 \,\big)\, \frac{43}{17}s^2 + \frac{8}{17}s^3 \,\big(\cancel{\frac{2367}{387}}\ \frac{387}{2367} \times \frac{43}{17} \leftarrow \frac{1}{R_4} \Rightarrow R_4 = \frac{1}{0.414}$$
$$\frac{43}{17}s^2 + \cancel{\frac{43}{17}}\, 0.414\, s^3$$

$$0.057 s^3 \,\big)\, 6.116s^2 + s^3 \,\big(107.3/s \leftarrow \frac{1}{C_5 s} \Rightarrow C_5 = \frac{1}{107.3}$$
$$6.116 s^2$$

$$s^3 \,\big)\, 0.057s^3 \,\big(0.057 \leftarrow \frac{1}{R_6} \Rightarrow R_6 = \frac{1}{0.057}$$
$$0.057s^3$$
$$0$$



25) An Impedance function has
i) simple poles at -1 & -4
ii) simple zeros at -2 & -5
iii) $Z(\infty) = 10\,\Omega$

Solution :

The $Z(s)$ can be formed as

$$Z(s) = H\,\frac{(s+2)\,(s+5)}{(s+1)\,(s+4)}$$

$$Z(0) = 10 = H\,\frac{(2)\,(5)}{(1)\,(4)} = \frac{10}{4} \Rightarrow H = 4$$

1. poles & zeros on negative real axis
2. Alternate poles and zeros

Hence it is an RC Impedance function

Foster-I form:

$$Z(s) = 4\,\frac{(s+2)\,(s+5)}{(s+1)\,(s+4)}$$

$$s^2+5s+4\,\overline{)\,4s^2+28s+40\,}\,(4$$
$$4s^2+20s+16$$
$$8s+24$$

$$\therefore Z(s) = 4 + \frac{8s+24}{(s+1)(s+4)}$$

$$Z(s) = \frac{A}{s+1} + \frac{B}{s+4} + 4$$

$$A = \cancel{(s+1)}\,\frac{8s+24}{\cancel{(s+1)}\,(s+4)}\Bigg|_{s=-1} \Rightarrow \frac{-8+24}{3} = \frac{16}{3}$$

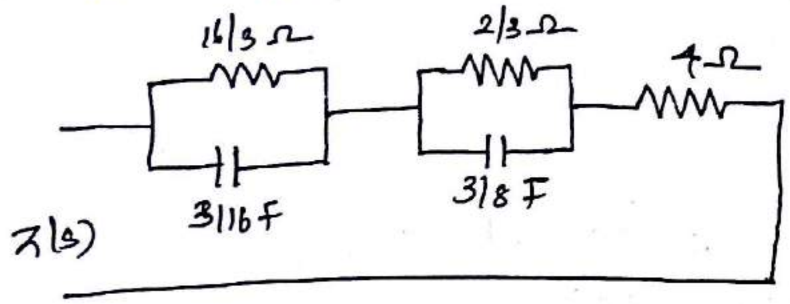
$$B = \cancel{(s+4)}\,\frac{8s+24}{(s+1)\,\cancel{(s+4)}}\Bigg|_{s=-4} \Rightarrow \frac{-32+24}{(-3)} = \frac{8}{3}$$

$$Z(s) = \frac{(16/3)}{s+1} + \frac{(8/3)}{s+4} + 4$$

$$\therefore C_0 = \frac{1}{P_0} = 0 \qquad R_\infty = H = 4\,\Omega$$

$$C_1 = \frac{1}{P_1} = \frac{1}{(16/3)} = \frac{3}{16}, \; P_1 = \frac{1}{\sigma_1 C_1} = \frac{1}{1\times(3/16)} = \frac{16}{3}$$

$$C_2 = \frac{1}{P_2} = \frac{1}{(8/3)} = \frac{3}{8}, \; P_2 = \frac{1}{\sigma_2 C_2} = \frac{1}{4\times(3/8)} = \frac{2}{3}$$



Foster II form:

$$Y(s) = \frac{(s+1)(s+4)}{4(s+2)(s+5)}$$

$$4s^2+28s+40 \,)\, s^2+5s+4 \,(\tfrac{1}{4}$$
$$\underline{s^2+7s+10}$$
$$\underline{-2s-6}$$

The values are negative, so $\frac{Y(s)}{s}$

$$\frac{Y(s)}{s} = \frac{(s+1)(s+4)}{4s(s+2)(s+5)}$$

$$\frac{Y(s)}{s} = \frac{A}{4s} + \frac{B}{s+2} + \frac{C}{s+5}$$

$$A = \cancel{4}s \left.\frac{(s+1)(s+4)}{\cancel{4}\cancel{s}(s+2)(s+5)}\right|_{s=0} \Rightarrow \frac{4}{4\times10} = \frac{1}{10}$$

$$B = (\cancel{s+2}) \left.\frac{(s+1)(s+4)}{4s(\cancel{s+2})(s+5)}\right|_{s=-2} \Rightarrow \frac{-1\times2}{4(-2)(3)} = \frac{1}{12}$$

$$C = (\cancel{s+5}) \left.\frac{(s+1)(s+4)}{4s(s+2)(\cancel{s+5})}\right|_{s=-5} \Rightarrow \frac{-4\times-1}{4(-5)\times-3} = \frac{1}{15}$$

$$\frac{Y(s)}{s} = \frac{(1/10)}{s} + \frac{(1/12)}{s+2} + \frac{(1/15)}{s+5}$$

$$Y(s) = \frac{1}{10} + \frac{(1/12)s}{s+2} + \frac{(1/15)s}{s+5}$$

$$\therefore R_0 = \frac{1}{10}$$

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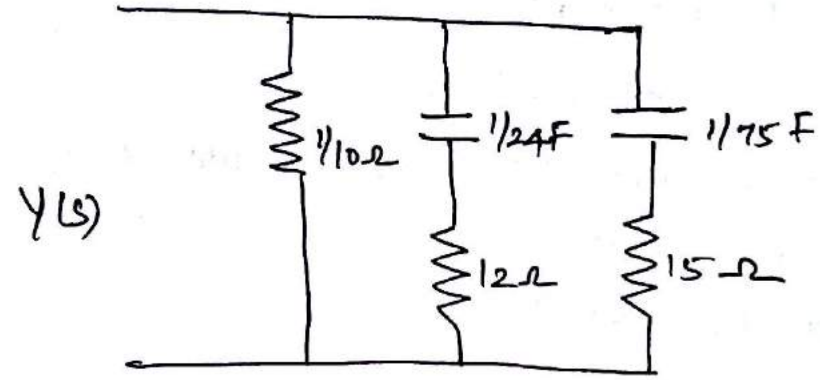
$$R_1 = \frac{1}{P_1} = \frac{1}{(1/12)} = 12,\quad C_1 = \frac{1}{\sigma_1 P_1} = \frac{1}{2\times 12} = \frac{1}{24}$$

$$R_2 = \frac{1}{P_2} = \frac{1}{(1/15)} = 15,\quad C_2 = \frac{1}{\sigma_2 P_2} = \frac{1}{5\times 15} = \frac{1}{75}$$



Cauer I form:

$$Z(s) = \frac{4s^2+28s+40}{s^2+5s+4}$$

$s^2+5s+4\,)\,4s^2+28s+40\,(\,4 \leftarrow R_1$

$\quad 4s^2+20s+16$

$\quad 8s+24\,)\,s^2+5s+4\,(\,\frac{s}{8} \leftarrow C_2$

$\quad\quad s^2+3s$

$\quad\quad 2s+4\,)\,8s+24\,(\,4 \leftarrow R_3$

$\quad\quad\quad 8s+16$

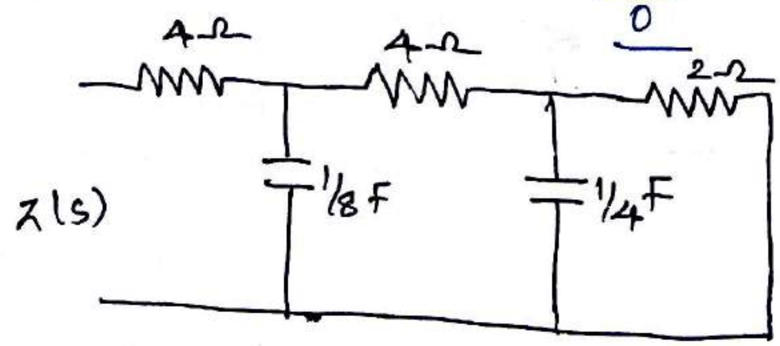
$\quad\quad\quad 8\,)\,2s+4\,(\,\frac{s}{4} \leftarrow C_4$

$\quad\quad\quad\quad 2s$

$\quad\quad\quad\quad 4\,)\,8\,(\,2 \leftarrow R_5$

$\quad\quad\quad\quad\quad 8$

$\quad\quad\quad\quad\quad 0$



Cauer II form:

$$Z(s) = \frac{40+28s+4s^2}{4+5s+s^2}$$

$$4+5s+s^2 \overline{\big)\, 40+28s+4s^2 \,}\big(\, 10$$
$$\underline{40+50s+10s^2}$$
$$-22s-6s^2$$

Since the terms are negative we use $Y(s)$.

$$Y(s) = \frac{4+5s+s^2}{40+28s+4s^2}$$

$40+28s+4s^2 \big)\, 4+5s+s^2 \,\big(\, \frac{1}{10} \leftarrow \frac{1}{R_1}$ $\quad R_1 = 10\Omega$

$\underline{4+2.8s+0.4s^2}$

$2.2s+0.6s^2 \big)\, 40+28s+4s^2 \,\big(\, \frac{18.2}{s} \leftarrow \frac{1}{C_2 s}$ $\quad B_2 = 0.055F$

$\underline{40+10.92s}$

$17.08s+4s^2 \big)\, 2.2s+0.6s^2 \,\big(\, 0.129 \leftarrow \frac{1}{R_3}$ $\quad R_3 = 7.75\Omega$

$\underline{2.2s+0.516s^2}$

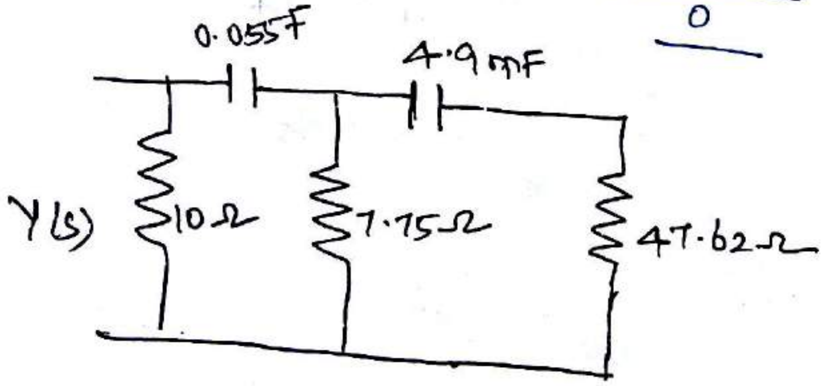
$0.084s^2 \big)\, 17.08s+4s^2 \,\big(\, \frac{203.4}{s} \leftarrow \frac{1}{C_4 s}$ $\quad C_4 = 4.9mF$

$\underline{17.08s}$

$4s^2 \big)\, 0.084s^2 \,\big(\, 0.021 \leftarrow \frac{1}{R_5}$ $\quad R_5 = 47.62\Omega$

$\underline{0.084s^2}$

$\underline{0}$



2b) Find which of the following are RL impedance functions. Synthesize one of the function in foster I and one cauer form.

i) $Z(s) = \frac{4(s+1)(s+3)}{s(s+2)}$ ii) $Z(s) = \frac{2(s+1)(s+3)}{(s+2)(s+6)}$

Solution:

i) $Z(s) = \frac{4(s+1)(s+3)}{s(s+2)}$

1. A pole at $s=0$
2. All poles and zeros lie on real negative axis
3. Alternate poles and zeros

This means that it is either an RC or RL impedance function.

Now we find residues at poles,

$$Z(s) = \frac{A}{s} + \frac{B}{s+2}$$

$$A = \cancel{s}\, \frac{4(s+1)(s+3)}{\cancel{s}(s+2)}\Big|_{s=0} \Rightarrow \frac{12}{2} = 6$$

$$B = \cancel{(s+2)}\, \frac{4(s+1)(s+3)}{s\cancel{(s+2)}}\Big|_{s=-2} \Rightarrow \frac{4(-1)(1)}{(-2)} = 2$$

Both the residues are positive. ∴ It is an RC impedance function.

ii) $Z(s) = \dfrac{2(s+1)(s+3)}{(s+2)(s+6)}$

1. All poles and zeros on real negative axis
2. Alternate poles and zeros $\therefore$ either RC or RL impedance

Residues as, $Z(s) = \dfrac{A}{s+2} + \dfrac{B}{s+6}$

$$A = \cancel{(s+2)} \frac{2(s+1)(s+3)}{\cancel{(s+2)}(s+6)}\Bigg|_{s=-2} \Rightarrow \frac{2(-1)(1)}{(4)} = \frac{-1}{2}$$

$$B = \cancel{(s+6)} \frac{2(s+1)(s+3)}{(s+2)\cancel{(s+6)}}\Bigg|_{s=-6} \Rightarrow \frac{2(-5)(-3)}{(-4)} = -\frac{15}{2}$$

Since both the residues are negative, it is an RL impedance function.

Foster - I form:

$$Z(s) = \frac{2s^2+8s+6}{s^2+8s+12}$$

$$
\begin{array}{r}
s^2+8s+12 \,\big)\, 2s^2+8s+6 \,\big(\, 2 \\
\underline{2s^2+16s+24} \\
-8s-18 \\
\hline
\end{array}
$$

The terms are negative, so $\dfrac{Z(s)}{s}$

$$\frac{Z(s)}{s} = \frac{2(s+1)(s+3)}{s(s+2)(s+6)} = \frac{A}{s} + \frac{B}{s+2} + \frac{C}{s+6}$$

$$A = \cancel{s} \frac{2(s+1)(s+3)}{\cancel{s}(s+2)(s+6)}\Bigg|_{s=0} = \frac{2(1)(3)}{(2)(6)} = \frac{1}{2}$$

$$B = \cancel{(s+2)} \frac{2(s+1)(s+3)}{s\cancel{(s+2)}(s+6)}\Bigg|_{s=-2} = \frac{2(-1)(1)}{(-2)(4)} = \frac{1}{4}$$

$$c = (s+6)\frac{2(s+1)(s+3)}{s(s+2)(s+6)}\Bigg|_{s=-6} \Rightarrow \frac{2(-5)(-3)}{(-6)(-4)} = \frac{5}{4}$$

$$\therefore \frac{Z(s)}{s} = \frac{(1/2)}{s} + \frac{(1/4)}{s+2} + \frac{(5/4)}{s+6}$$

$$Z(s) = (1/2) + \frac{(1/4)s}{s+2} + \frac{(5/4)s}{s+6}$$

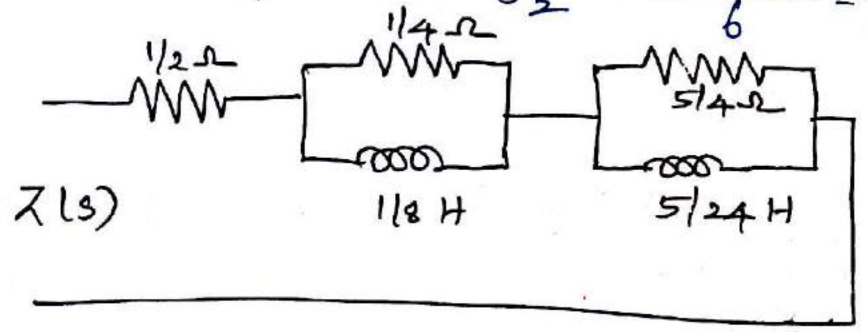
$$Z(s) = P_0 + \frac{P_1 s}{s+\sigma_1} + \frac{P_2 s}{s+\sigma_2} + H$$

$H = 0 = L_\alpha$

$P_0 = R_0 = \frac{1}{2}$

$R_1 = P_1 = \frac{1}{4}$, $L_1 = \frac{R_1}{\sigma_1} = \frac{(1/4)}{2} = \frac{1}{8}$

$R_2 = P_2 = \frac{5}{4}$, $L_2 = \frac{R_2}{\sigma_2} = \frac{(5/4)}{6} = \frac{5}{24}$



Cauer I form:

$$Z(s) = \frac{2s^2+8s+6}{s^2+8s+12}$$

$$s^2+8s+12 \,\big)\, 2s^2+8s+6 \,(\, 2$$
$$2s^2+16s+24$$
$$\overline{-8s-18}$$

Since there is negative term in $Z(s)$, we use $Y(s)$

$$Y(s) = \frac{s^2+8s+12}{2s^2+8s+6}$$

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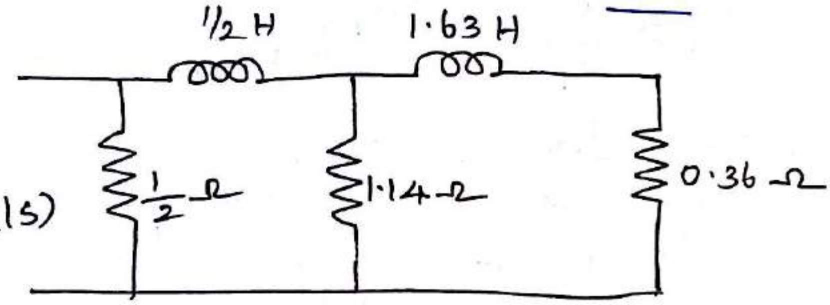
$$2s^2+8s+6 \,)\, s^2+8s+12 \,(\, \tfrac{1}{2} \leftarrow R_1$$
$$s^2+4s+3$$
$$4s+9 \,)\, 2s^2+8s+6 \,(\, \tfrac{s}{2} \leftarrow L_2$$
$$2s^2+4.5s$$
$$3.5s+6 \,)\, 4s+9 \,(\, 1.14 \leftarrow R_3$$
$$4s+6.85$$
$$2.15 \,)\, 3.5s+6 \,(\, 1.63s \leftarrow L_4$$
$$3.5s$$
$$6 \,)\, 2.15 \,(\, 0.36 \leftarrow R_5$$
$$2.15$$
$$0$$



Foster II form:

$$Y(s) = \frac{(s+2)(s+6)}{2(s+1)(s+3)} = \frac{s^2+8s+12}{2s^2+8s+6}$$

$$2s^2+8s+6 \,\big)\, s^2+8s+12 \,\big(\, \frac{1}{2}$$

$$\underline{s^2+4s+3}$$

$$\underline{4s+9}$$

$$\therefore Y(s) = \left(\frac{1}{2}\right) + \frac{4s+9}{2(s+1)(s+3)}$$

$$Y(s) = \frac{1}{2} + \frac{A}{s+1} + \frac{B}{s+3}$$

$$A = \cancel{(s+1)} \frac{4s+9}{2\cancel{(s+1)}(s+3)} \Bigg|_{s=-1} \Rightarrow \frac{5}{4}$$

$$B = \cancel{(s+3)} \frac{4s+9}{2(s+1)\cancel{(s+3)}} \Bigg|_{s=-3} \Rightarrow \frac{-3}{-4} = \frac{3}{4}$$

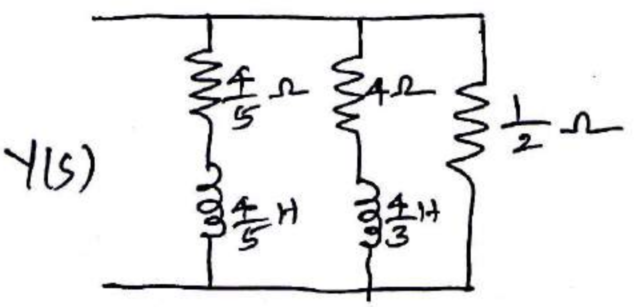
$$Y(s) = \frac{1}{2} + \frac{(5/4)}{s+1} + \frac{(3/4)}{s+3}$$

$$Y(s) = H + \frac{P_1}{s+\sigma_1} + \frac{P_2}{s+\sigma_2}$$

$$R_\alpha = H = \frac{1}{2}\,\Omega \quad , L_0 = 0$$

$$R_1 = \frac{\sigma_1}{P_1} = \frac{1}{(5/4)} = \frac{4}{5} \,,\; L_1 = \frac{1}{P_1} = \frac{1}{(5/4)} = \frac{4}{5}$$

$$R_2 = \frac{\sigma_2}{P_2} = \frac{3}{(3/4)} = 4 \,,\; L_2 = \frac{1}{P_2} = \frac{1}{(3/4)} = \frac{4}{3}$$



Cauer II form:

$$Z(s) = \frac{6+8s+2s^2}{12+8s+s^2}$$

$$12+8s+s^2 \overline{)\, 6+8s+2s^2} \left(\frac{1}{2} \leftarrow \frac{1}{R_1} \Rightarrow R_1 = 2 \right.$$
$$\underline{6+4s+0.5s^2}$$

$$4s+1.5s^2 \overline{)\, 12+8s+s^2} \left(\frac{3}{s} \leftarrow \frac{1}{L_2} \Rightarrow L_2 = \frac{1}{3} \right.$$
$$\underline{12+4.5s}$$

$$3.5s+s^2 \overline{)\, 4s+1.5s^2} \left(\frac{4}{3.5} \leftarrow \frac{1}{R_3} \Rightarrow R_3 = \frac{3.5}{4} \right.$$
$$\underline{4s+1.14s^2}$$

$$0.36s^2 \overline{)\, 3.5s+s^2} \left(\frac{3.5}{0.36s} \leftarrow \frac{1}{L_4} \Rightarrow L_4 = \frac{0.36}{3.5} \right.$$
$$\underline{3.5s}$$

$$s^2 \overline{)\, 0.36s^2} \left(0.36 \leftarrow \frac{1}{R_5} \Rightarrow R_5 = \frac{1}{0.36} \right.$$
$$\underline{0.36s^2}$$
$$\underline{0}$$