# UNIT 5

## ANALYSIS OF SIGNALS

The topics are

* Sampling of CT Signals
  - Sampling Theorem
  - Effect of under sampling
  - Aliasing
  - Reconstruction of CT Signals from samples
* Fourier Series Representation of DT periodic signals
  - properties
* Fourier transform Representation of DT aperiodic signals
  - properties

# Sampling of continuous time signal

## Sampling:

Sampling is a process of continuous time signal into discrete time signal.

After sampling, the signal is defined at discrete instants of time and the time interval between two subsequent sampling instants is called Sampling Interval.

The important factor is the signal sampling rate must be kept sufficiently high so that the original signal can be reconstructed from its samples.

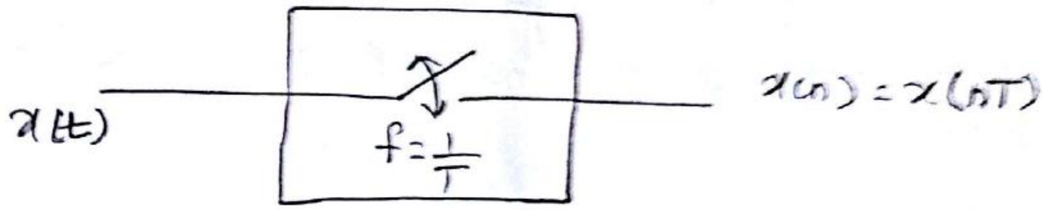
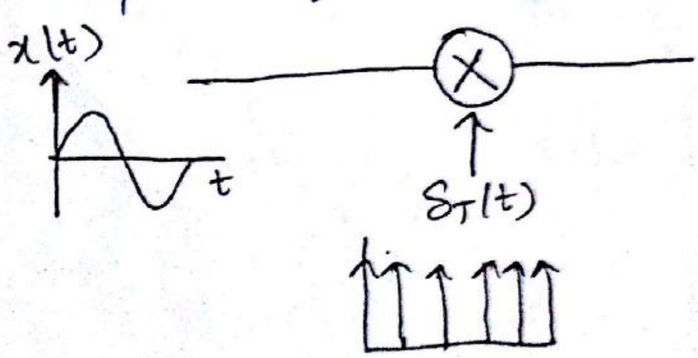
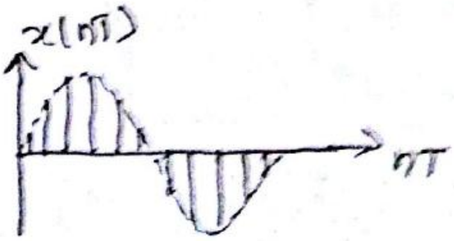



fig 1(a)

* The switch is closed for a very short interval of time $\tau$, during which the signal is available at the output as shown in fig 1(a).
* If the input is $x(t)$ then the output is $x(nT)$, $n = 0, \pm 1, \pm 2, \ldots$ and $x(nT)$ is called the sampled sequence of $x(t)$.



Where T is the time interval between successive samples and the sampling frequency is $f_s = \frac{1}{T}$ Hz.

* The discrete time signal is

$$x_s(t) = x(t)\ \delta_T(t) \quad \text{—} \quad (1)$$

where, $\delta_T(t) = \sum_{n=-\alpha}^{\alpha} \delta(t-nT)$ — (2)

Sub (2) in (1)

$$x_s(t) = \sum_{n=-\alpha}^{\alpha} x(nT)\ \delta(t-nT) \quad \text{—} \quad (3)$$

which is a sequence of impulses at $t = nT$ with weight $x(nT)$.

Thus $x_s(t)$ can be considered as a continuous time representation of the discrete time sequence $x(nT)$.

Apply fourier transform to eq (3)

$$F[x_s(t)] = F\left[\sum_{n=-\alpha}^{\alpha} x(nT)\ \delta(t-nT)\right]$$

$$F[x_s(t)] = \sum_{n=-\alpha}^{\alpha} x(nT)\ F[\delta(t-nT)] \quad \text{—} \quad (4)$$

$$F[\delta(t-nT)] = \int_{-\alpha}^{\alpha} \delta(t-nT)\ e^{-j\omega t}\ dt$$

$$= e^{-j\omega nT} \quad \text{—} \quad (5)$$

$$\therefore X(e^{j\omega}) = F[x_s(t)] = \sum_{n=-\alpha}^{\alpha} x(nT)\ e^{-j\omega nT}$$

$$\delta_T(t) = \sum_{m=-\infty}^{\infty} \delta(t-nT)$$

$$\delta_T(t) = \frac{1}{T}\sum_{m=-\infty}^{\infty} e^{jm\Omega_0 t} \quad \text{—— (7)}$$

where $\Omega_0 = \frac{2\pi}{T}$

Sub (7) in (1),

$$x_s(t) = x(t)\left[\frac{1}{T}\sum_{m=-\infty}^{\infty} e^{jm\Omega_0 t}\right]$$

$$x_s(t) = \frac{1}{T}\sum_{m=-\infty}^{\infty} x(t)\, e^{jm\Omega_0 t} \quad \text{— (8)}$$

$$F\left[x_s(t)\right] = \frac{1}{T}\sum_{m=-\infty}^{\infty} F\left[x(t)\, e^{jm\Omega_0 t}\right] \quad \text{— (9)}$$

If $F\left[x(t)\right] = X(j\Omega)$ then,

$$F\left[x(t)\, e^{jm\Omega_0 t}\right] = X\left(j(\Omega - m\Omega_0)\right) \quad \text{— (10)}$$

Sub (10) in (9)

$$F\left[x_s(t)\right] = \frac{1}{T}\sum_{m=-\infty}^{\infty} X\left[j(\Omega - m\Omega_0)\right] \quad \text{— (11)}$$

$$F\left[x_s(t)\right] = \frac{1}{T}\sum_{m=-\infty}^{\infty} X\left[j\left(\Omega - \frac{2\pi m}{T}\right)\right] \quad \text{— (12)}$$

Thus the fourier transform of the sampled signal is given by an infinite sum of shifted replicas of the fourier transform of the original signal.

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## Aliasing Effect:

When we produce the sequence $x(n)$ by sampling $x_a(t)$, we want to ensure that all the information in the original signal is retained in the samples.

Now we determine the condition under when there is no information loss.

* Now consider the signal $x(t)$ is band limited to $f_m$. That is the highest frequency is $f_m$. Then,

$$X(j\Omega) = 0 \quad \text{for } |\Omega| > \Omega_m \quad (\Omega_m = 2\pi f_m)$$

i) from eq (12),

$X\left(j\left(\Omega - \frac{2\pi m}{T}\right)\right)$ is the shifting of $X(j\Omega)$ from $\Omega = 0$ to $\Omega = \frac{2\pi m}{T}$.

Hence $X_s(j\Omega)$ is the sum of shifted replicas of $\frac{X(j\Omega)}{T}$ centering at $\frac{2\pi m}{T}$, $m = 0, \pm1, \pm2, \ldots$ . fig shows the plot of $\frac{X(j\Omega)}{T}$ for various values of $\frac{\pi}{T}$.

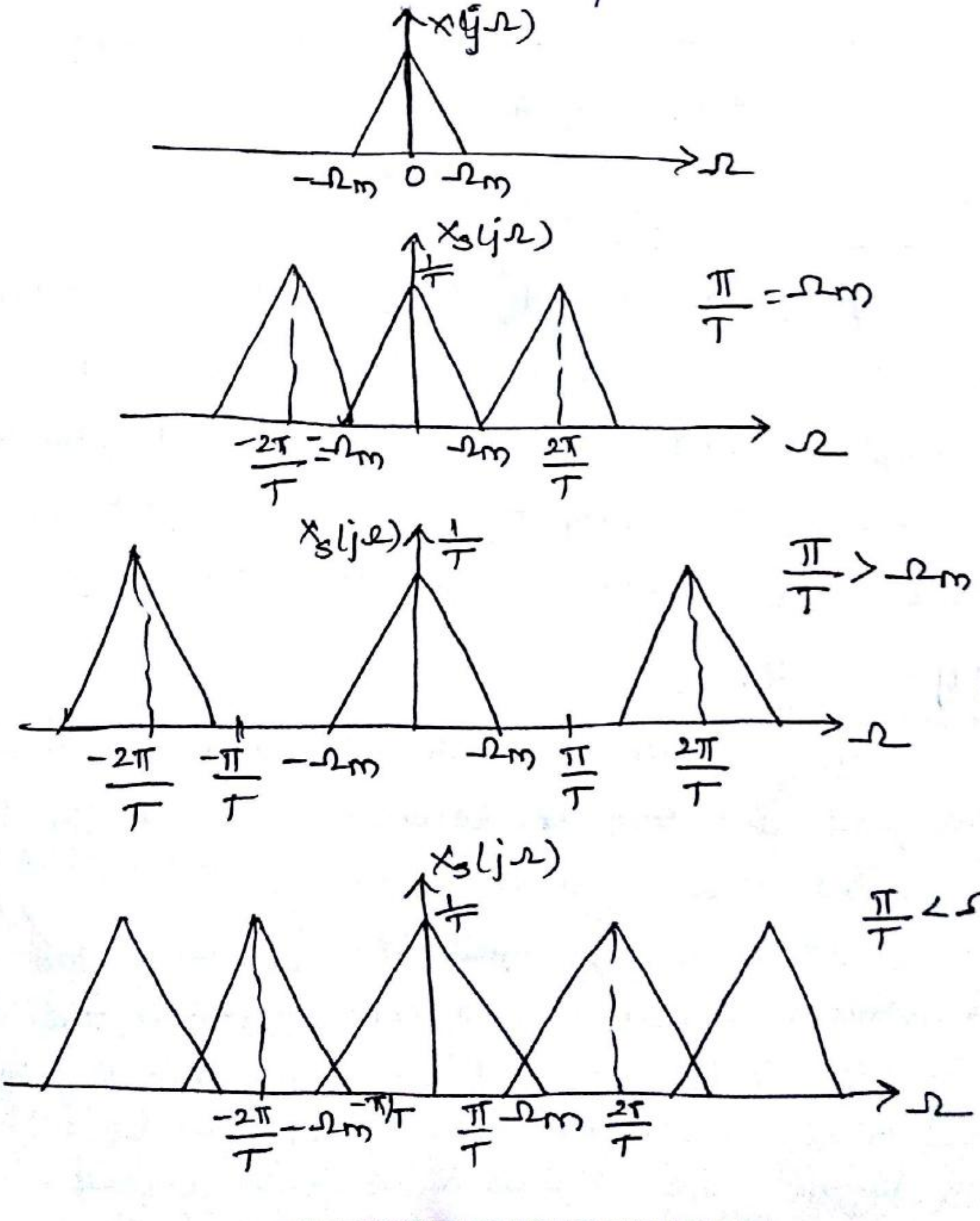
ii) If $\frac{\pi}{T} > \Omega_m$, the replicas will not overlap & as a result, the frequency spectrum of $T$, $X_s(j\Omega)$ in the frequency range $\left(\frac{-\pi}{T}, \frac{\pi}{T}\right)$ is identical to $X(j\Omega)$. Thus the frequency spectrum of $X(j\Omega)$ can be recovered from $X_s(j\Omega)$ by using a low pass filter with has sharp cut off at $\Omega = \frac{\pi}{T}$.

iii) The same explanation for $\frac{\pi}{T} = \Omega_m$.

iv) If $\frac{\pi}{T} < \Omega_m$, the successive frequency replicas will overlap as shown in the fig. and as a result, the frequency spectrum $X(j\Omega)$ will not be recovered from the frequency spectrum of $TX_s(j\Omega)$.

* The super imposition of high frequency component on the low frequency is known as frequency aliasing.

* Because of aliasing, the spectrum $X(j\Omega)$ is no longer recoverable from the spectrum of $X_s(j\Omega)$.



* from fig, the aliasing error can be prevented if the highest frequency component $\Omega_m$ in the signal $x(t)$ is less than or equal to $\frac{\pi}{T}$, (ie $\frac{\pi}{T} \geq \Omega_m$)

* If $f_s = \frac{1}{T}$ then,

$$\pi f_s \geq \Omega_m$$
$$\pi f_s \geq 2\pi f_m$$
$$f_s \geq 2 f_m$$

* To avoid aliasing the sampling frequency must be greater than twice the highest frequency present in the signal.

## Sampling Theorem:

A band limited signal $x(t)$ with $X(j\Omega) = 0$ for $|\Omega| > \Omega_m$ is uniquely determined from its samples $x(nT)$, if the sampling frequency $f_s \geq 2f_m$ (ie) sampling frequency must be atleast twice the highest frequency present in the signal.

## Nyquist Rate:

The minimum rate at which a signal can be sampled and still be reconstructed from its samples is called the Nyquist rate.

It is always equal to $2f_m$ where $f_m$ is the maximum frequency component present in the signal.

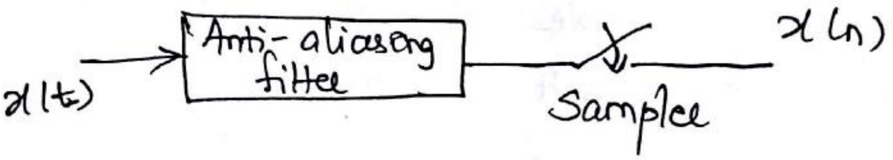
A signal sampled at greater than Nyquist rate is said to be oversampled and a signal sampled at less than its Nyquist rate is said to be undersampled.

## Anti Aliasing Filter:

Sampling theorem states that a signal can be perfectly reconstructed if it is band limited. In practice, communication signals have frequency spectra consisting of low frequency components as well as high frequency noise components.

When the signal is sampled, with sampling frequency fs, all signals with frequency large higher than $\frac{\Omega_s}{s}$ appear a signal frequencies between 0 and $\frac{\Omega_s}{2}$ creating aliasing.

Therefore to avoid aliasing errors caused by the undesired high frequency signals, an analog lowpass filter called an antialiasing filter is used prior to sampler as shown in the figure.




Definition: A filter that is used to reject high frequency signals before it is sampled to reduce the aliasing is called an antialiasing filter.

## Signal Reconstruction:

The signal can be reconstructed from its samples if the sampling rate is above Nyquist rate.

Let the sampled signal $x_s(t)$ with sampling period T second is

$$x_s(t) = \sum_{n=-\infty}^{\infty} x(nT)\,\delta(t-nT) \quad \text{— (1)}$$

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The fourier transform of $x_s(t)$ is

$$X(e^{j\omega}) = \frac{1}{T} \sum_{m=-\infty}^{\infty} X\left[j\left(\Omega - \frac{2\pi m}{T}\right)\right] \quad - (2)$$

The frequency spectrum $X(e^{j\omega})$ consists of an infinite sum of shifted replicas of the frequency spectrum of the original signal.

* To get the original spectrum the signal $x_s(t)$ is to be passed through a reconstruction (low-pass filter with an impulse response $h(t)$.

$$x_s(t) \longrightarrow \boxed{h(t)} \longrightarrow x(t)$$

* The original signal can be obtained by convolving the sampled signal $x_s(t)$ and the impulse response $h(t)$ of the filter.

$$x(t) = x_s(t) * h(t)$$

$$= \left[\sum_{n=-\infty}^{\infty} x(nT)\, \delta(t-nT)\right] * h(t)$$

$$= \int_{-\infty}^{\infty} \sum_{n=-\infty}^{\infty} x(nT)\, \delta(\lambda - nT)\, h(t-\lambda)\, d\lambda$$

$$\because \delta(\lambda - nT) = 1$$
$$0$$

$$= \sum_{n=-\infty}^{\infty} x(nT) \int_{-\infty}^{\infty} \delta(\lambda - nT)\, h(t-\lambda)\, d\lambda$$

$$= \sum_{n=-\infty}^{\infty} x(nT)\, h(t-nT) \quad - 3$$

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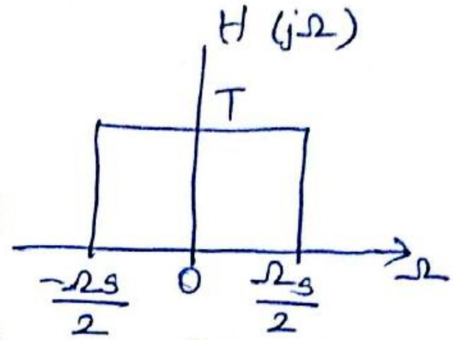
* The frequency response of the lowpass filter can be expressed by

$$H(j\Omega) = T \quad \text{for } |\Omega| \leq \frac{\Omega_s}{2}$$

$$0 \quad |\Omega| > \frac{\Omega_s}{2} \quad - (4)$$




* The Impulse response of the lowpass filter is

$$h(t) = \frac{1}{2\pi}\int_{-\frac{\Omega_s}{2}}^{\frac{\Omega_s}{2}} H(j\Omega)\, e^{j\Omega t}\, d\Omega$$

$$= \frac{T}{2\pi}\int_{-\frac{\Omega_s}{2}}^{\frac{\Omega_s}{2}} e^{j\Omega t}\, d\Omega$$

$$= \frac{T}{\pi t} \sin \frac{\Omega_s t}{2}$$

$$= \frac{\pi}{\pi t} \sin \frac{\pi t}{T}$$

$$= \frac{T}{\pi} \frac{\sin \frac{\pi t}{T}}{t} \quad - (5)$$

$$h(t-nT) = \frac{T}{\pi} \frac{\sin \frac{\pi}{T}(t-nT)}{(t-nT)} \quad - (5)$$

sub eq (5) in (3)

$$x(t) = \sum_{n=-\infty}^{\infty} x(nT)\, \text{sinc}\, \frac{\pi}{T}(t-nT)$$

* The response $x(t)$ is not suitable for practical implementation.

* To reconstruct the signal at time $t = t_0$, we must know all the samples values including those for $nT > t_0$.

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* $x(t_0)$ requires the knowledge of future values of $x(nT)$. So it represents a non-causal system. Hence this type of reconstruction is not suitable for real time applications.

* To overcome this, digital to analog converter is used, it produces an output voltage that is proportional to the value of the binary word.

* the output of DAC is applied to an interpolator which converts the output samples of DAC into an analog signal.


Zero order hold:

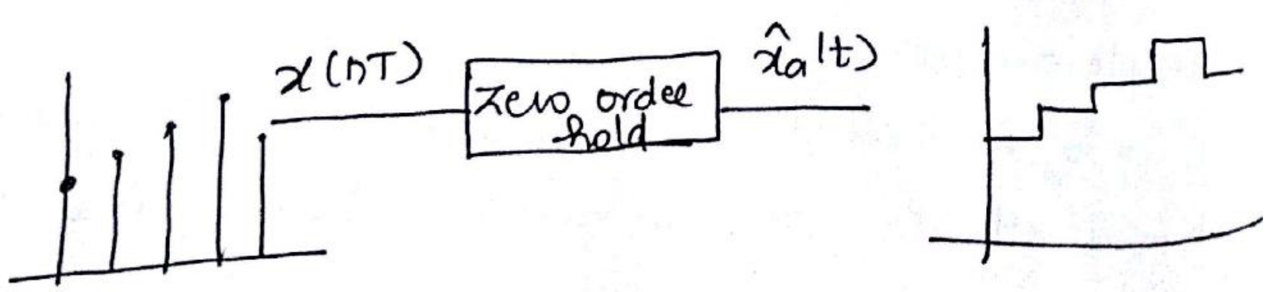
* Most widely used interpolator is zero order hold.

* In this, a given sample is hold for an interval until the next sample is received.

$$\hat{x}_a(t) = x(n) \quad \text{for } nT \le n \le (n+1)T \quad — (1)$$

$$n = 0, 1, 2 \ldots$$

$$\hat{x}_a(t) = x(0) \quad \text{for } 0 \le t < T$$
$$= x(T) \quad T \le t < 2T \quad — (2)$$
$$= x(2T) \quad 2T < t < 3T$$



* The impulse response of a zero order hold is

$$h(t) = 1 \quad 0 \le t \le T \qquad — (3)$$
$$\quad 0 \quad \text{otherwise}$$

**Transfer function of a zero order hold**

* The output of zero order hold is expressed interms of $h(t)$.

* The output $\hat{x}_a(t)$ is the convolution of $x(nT)$ and $h(t)$.

$$\hat{x}_a(t) = \sum_{n=-\infty}^{\infty} x(nT)\, h(t-nT) \qquad — (4)$$

$h(t)$ in eq (3) can be expressed as,

$$h(t) = u(t) - u(t-T) \qquad — (5)$$

In general,

$$h(t-nT) = u(t-nT) - u(t-(n+1)T) \qquad — (6)$$

Sub. (6) in (4)

$$\hat{x}_a(t) = \sum_{n=-\infty}^{\infty} x(nT)\left[u(t-nT) - u(t-(n+1)T)\right]$$

Taking Laplace transform on both sides,

$$L[\hat{x}_a(t)] = \sum_{n=-\infty}^{\infty} x(nT)\, L\{u(t-nT) - u[t-(n+1)T]\}$$

$$= \sum_{n=-\infty}^{\infty} x(nT)\left[\frac{e^{-snT}}{s} - \frac{e^{-s(n+1)T}}{s}\right]$$

$$= \frac{1-e^{-sT}}{s} \sum_{n=0}^{\infty} x(nT)\, e^{-snT}$$

$$= \left[\frac{1-e^{-ST}}{s}\right] X^*(s)$$

$$\frac{X_a(s)}{X^*(s)} = \frac{1-e^{-ST}}{s}$$

* The output of a zero order hold consists of higher order harmonics. To remove these harmonics the output of zero order hold is applied to a low pass filter as shown in the figure.

* This filter is often called a smoothing filter because it tends to smooth the corners on the step approximations generated by the ZOH.

Problems:

1. A signal having a spectrum ranging from dc to 10 KHz is to be sampled and converted into discrete form. What is the minimum number of samples per second that must be taken to ensure recovery

Solution:

Given $f_m = 10\,KHz$

$f_s = 2f_m$

$= 20{,}000$ samples/sec

2) A signal $x(t) = \text{sinc}(150\pi t)$ is sampled at a rate of a) 100 Hz b) 200 Hz c) 300 Hz. For each of these three cases, can you recover the signal $x(t)$ from the sampled signal?

Solution:

$$2\pi f_m = 150\pi$$

$$f_m = 75 \text{ Hz}$$

Nyquist rate $= 2f_m = 150$ Hz

a) In case a, 100 Hz < 150 Hz ⇒ Not reconstruction

b) In case b & c, 200 Hz, 300 Hz > 150 Hz ⇒ Signal can be reconstructed